Twisted Second Moments and Explicit Formulae of the Riemann Zeta-Function

Dissertation

zur

Erlangung der naturwissenschaftlichen Doktorwürde
(Dr. sc. nat.)

vorgelegt der

Mathematisch-naturwissenschaftlichen Fakultät

der

Universität Zürich

von

Nicolas MARTINEZ ROBLES

von

Madrid (Spanien)

Promotionskomitee

Prof. Dr. Alberto Cattaneo (Vorsitz und Leitung)
Prof. Dr. Ashkan Nikeghbali
Prof. Dr. Joseph Ayoub
Prof. Dr. Bruce Berndt (Gutachter, Urbana, Illinois)

Zürich 2015

Declaration of Authorship

I, Nicolas Martinez Robles, declare that this thesis titled, 'Twisted Second Moments and Explicit Formulae of the Riemann Zeta-Function' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ______________________________

Date: ______________________________

"... Davenport was the first mathematician I met who had a group of young research students around him and regularly supplied them with problems... I told him my ideas about the Siegel conjecture and he encouraged me to continue my efforts. At that time I had tentatively decided to switch my activities from mathematics to physics... I thought that it would be more exciting to solve one of the basic mysteries of nature than to continue proving theorems that were of interest only to a small coterie of number theorists. But Davenport's friendliness tempted me to stay with mathematics. I decided to launch an all out attack on the Siegel conjecture and to let the result determine my future. If I succeeded in proving it, I would be a mathematician. If I failed, I would be a physicist. After three months of intensive work, I admitted failure. I would after all be a physicist... ... It was easy for me to switch from mathematics to physics, because both number theory and physics are branches of applied mathematics. I define a pure mathematician to be somebody who creates mathematical ideas, and I define an applied mathematician to be somebody who uses existing mathematical ideas to solve problems. According to this definition, I was always an applied mathematician, whether I was solving problems in number theory or in physics... The main difference between number theory and physics is that in number theory the experimental data are more accurately known. In recent years the increasing use of computers has made number theory more than ever an experimental science..."

Freeman Dyson [Dys96], quoted from [Boh05].

UNIVERSITÄT ZÜRICH

Abstract

Mathematisch-naturwissenschaftlichen Fakultät
Universität Zürich

Doctor of Philosophy

Twisted Second Moments and Explicit Formulae of the Riemann Zeta-Function

by Nicolas MARTINEZ ROBLES

Verschiedene Aspekte, die analytische Zahlentheorie und die Riemann zeta-Funktion verbinden, werden erweitert. Dies beinhaltet:

1. explizite Formeln, die eine Verbindung zwischen der Möbiusfunktion und den nicht-trivialen Nullstellen der zeta-Funktion herstellen;
2. verallgemeinerte Resultate über Summen von Ramanujan Summen;
3. neue Resultate über die Kombinationen von Riemann Ξ-Funktionen auf beschränkten vertikalen Verschiebungen und ihre Nullstellen auf der kritischen Geraden;
4. Verallgemeinerung der Moment Integrale der Riemann Ξ-Funktion;
5. asymptotische Näherungen der durchschnittlichen Quadrate der Produkte der Riemann ζ-Funktion und neuer Dirichlet Polynome;
6. zeta Regularisierung auf Tori und einen neuen Beweis der Chowla-Selberg Formel.

Several aspects connecting analytic number theory and the Riemann zeta-function are studied and expanded. These include:

1. explicit formulae relating the Möbius function to the non-trivial zeros of the zeta function;
2. generalized results on sums of Ramanujan sums;
3. new results on the combinations of Riemann Ξ-functions on bounded vertical shifts and their zeros on the critical line;
4. a generalization of moment integrals involving the Riemann Ξ-function;
5. asymptotics for the mean square of the product of the Riemann ζ-function and new Dirichlet polynomials;
6. zeta regularization on tori and a new proof of the Chowla-Selberg formula.

Acknowledgements

Most acknowledgments in a doctoral thesis begin by thanking the supervisor. My case is certainly no different. Therefore, I should first like to thank my advisers Alberto Cattaneo and Ashkan Nikeghbali for accepting the supervision of this thesis. Without their patience, their encouragement and the freedom they have provided, none of this would have taken place. It is to them that I owe this dissertation.

Some truly fantastic memories of our time together come to my mind, e.g.: sprinting out of the classrooms in Les Houches to go skiing in Montblanc with Alberto, or driving around southern California with Ashkan.

I have been extremely lucky and fortunate to have had such great collaborators. I have learned much from working with them. I would like to thank Atul Dixit for agreeing to see me at Tulane University. What started casually as an exchange of ideas turned out to be a most profitable collaboration with him, Arindam Roy and Alexandru Zaharescu at UIUC. Arindam was kind enough to host me in Champaign several times. I have but the fondest memories of our time together as well as of our road trips to Washington DC and Kentucky. Working with Alexandru Zaharescu is nothing short of a privilege and I am extremely grateful to him for supporting my application for the J L Doob Research Assistant Professor position at UIUC.

Moreover, I wish to thank Emilio Elizalde, Klaus Kirsten, Patrick Kühn and Floyd Williams for having agreed to work with me.

Always an essential part of research, I am grateful to my research grants. In particular, I would like to acknowledge UZH for providing the Forschungskredit.

Hung Bui and Brian Conrey introduced me to the problem of the percentage of the zeros of the Riemann zeta-function on the critical line. Brian Conrey invited me to spend a week with him in Palo Alto, CA while I was visiting Ashkan on sabbatical at UC Irvine. Hung Bui has gone out of his way to answer my questions. Thank *you*!

Moreover, the week I spent in Mexico City with Jean Bertoin for the 7th School of Probability and Stochastic Processes was fantastic. Paul-Oliver Dehaye has been a great source of advice and guidance. Maksym Radziwill was an encouraging friend.

The referees of this thesis provided extremely useful comments, remarks and corrections which have greatly increased the quality of the manuscript.

I would like to thank all other group members - professors, postdocs, grad students and staff alike - who have contributed to the running of UZH and who have created a stimulating academic and social atmosphere at the Institüt für Mathematik.

Lastly, my family, my flatmate, my dear one and my friends have been a constant source of motivation.

J N M R

Contents

List of Figures

Symbols

$\mathbb{C}^*$	The complex plane with the origin excluded (the multiplicative group of complex numbers).
$\mathbb{H}$	The Poincaré space $\{\tau = \tau_1 + i\tau_2,\ \tau_1 \in \mathbb{R},\ \tau_2 > 0\}$.
$(\mathbb{Z}/q\mathbb{Z})^*$	The group of coprime residue classes mod q.
$\mathrm{SL}(2,\mathbb{Z})$	The special linear group, i.e. the group under matrix multiplication of 2×2 matrices over $\mathbb{Z}$ with determinant equal to 1.
$\mathcal{P}$	The set of prime numbers $\{2, 3, 5, 7, 11, \cdots\}$.
$\mathcal{C}^\infty$	The space of infinitely differentiable functions.
$\mathcal{L}(-\infty,\infty)$	The space of square integrable functions.
$\mathcal{S}$	The Selberg class defined in Chapter 1.
$K(\omega,\alpha)$	A class of functions defined in Chapter 1.
$\clubsuit(\omega,\lambda,\beta)$	A class of functions defined in Chapter 4.
$\spadesuit(\eta,\omega)$	A class of functions defined in Chapter 4.
$h,\ j,\ k,\ l,\ m,\ n, \cdots$	Natural numbers (positive integers).
$p,\ p_1,\ p_2,\ q, \cdots$	Prime numbers.
$A,\ B,\ C_1,\ C_2, \cdots$	Absolute constants (not necessarily the same at each occurrence in a proof).
B_1	In Chapter 2 only, this will denote Mertens constant.
$\delta,\ \delta_0,\ \varepsilon,\ \varepsilon_0,\ \eta, \cdots$	Arbitrarily small positive reals (not necessarily the same at each occurrence in a proof).
γ, γ_0	Euler's constant, defined by $\gamma = -\Gamma'(1)$.

$\gamma_1, \gamma_2, \cdots$	The Stieltjes constants. These appear in the Laurent expansion $\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\gamma_n(s-1)^n$.
$B_n, B_n(x)$	The Bernoulli numbers and Bernoulli polynomials.
$t,\ x,\ y, \cdots$	Real variables.
$s,\ u,\ w,\ z, \cdots$	Complex variables.
$s = \sigma + it$	where σ and t are reals.
τ	This is $\tau = \tau_1 + i\tau_2$ with $\tau_1 \in \mathbb{R}$ and $\tau_2 > 0$.
$\operatorname*{res}_{s=s_0} f(s)$	residue of $f(s)$ at $s = s_0$.
$\int_{(c)} f(s)ds$	$\int_{c-i\infty}^{c+i\infty} f(s)ds$.
$n(\gamma, a)$	The winding number of a closed curve γ with respect to the point a.
$\chi(n),\ \chi(s)$	In Chapter 1: a Dirichlet character, i.e. a homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ In Chapter 5: it is the function defined by $\zeta(s) = \chi(s)\zeta(1-s)$, so that $\chi(s) = (2\pi)^2/(2\Gamma(s)\cos(\pi s/2))$.
$\mathfrak{h}$	This is parity of the character χ, i.e. $\mathfrak{h} = 0$ if χ is even and $\mathfrak{h} = 1$ if χ is odd.
$s(h,k)$	This denotes a Dedekind sum defined in Chapter 6.
$\tau(\chi)$	A Gauss sum associated to a character χ mod q. It is defined by $\tau(\chi) = \sum_{m=1}^{q} \chi(m)e^{2\pi i m/q}$.
$\mu(n)$	This is the Möbius function. It is defined as $\mu(n) = (-1)^k$ if $n = p_1 \cdots p_k$ (the p_j's being different primes), and zero otherwise. Also $\mu(1) = 1$. In Chapter 3, this denotes the Lebesgue measure.
$M(x)$	This is the Mertens function $M(x) = \sum_{n \le x} \mu(n)$.
$\mu_m(n)$	These are the coefficients of the Dirichlet series $1/\zeta^m(s)$ for $m = 1, 2, 3, \cdots$ and for $\mathrm{Re}(s) > 1$. In particular $\mu_1(n) = \mu(n)$.
$\Lambda(n)$	This is the von Mangoldt function defined by $\Lambda(n) = \log p$ if $n = p^k$ and zero otherwise.
$d_k(n)$	This is the number of ways an integer n can be written as a product of $k \ge 2$ fixed factors. Also $d_1(n) = 1$, and

	$d_2(n) = d(n)$ denotes the number of divisors of n.
$\nu_p(n)$	This denotes the number of different prime factors of n. For simplicity in Chapter 5, we set $\nu_p(n) = n'$.
$\sigma_v(n)$	This is the sum of the v^{th} powers of the divisors of n.
(a, b)	Greatest common divisor of a and b.
$(h, q^\beta)_\beta = 1$	Means that h ranges over the non-positive integers less than q^β such that h and q^β have no common β-th divisor other than 1.
$c_q(n)$	The Ramanujan sum defined by $\sum_{(h,q)=1} e^{2\pi inh/q}$, the sum taken over a reduced residue system mod q.
$c_{q,\beta}(n)$	The Cohen-Ramanujan sum defined by $\sum_{(h,q^\beta)_\beta=1} e^{2\pi inh/q^\beta}$, the sum taken over a reduced residue system mod q.
$C_{k,\beta}(x, y)$	This is $\sum_{n\le y}(\sum_{q\le x} c_{q,\beta}(n))^k$.
H_r^n	The n^{th} harmonic number of order r defined by $H_r^n = \sum_{i=1}^r 1/i^n$.
$\zeta(s), \zeta_R(s)$	The Riemann-zeta function defined by $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ for $\mathrm{Re}(s) > 1$, and otherwise by analytic continuation.
$L(s, \chi)$	A Dirichlet L-function defined by $L(s, \chi) = \sum_{n=1}^\infty \chi(n)n^{-s}$ for $\mathrm{Re}(s) > 1$, and otherwise by analytic continuation.
$L(s)$	An L-function in the Selberg class $\mathcal{S}$.
$\zeta_M(s)$	A spectral zeta function associated with a smooth compact manifold M with Riemannian metric g.
$E^*(s, \tau)$	The non-holomorphic Eisenstein series $\sum_{(m,n)\in\mathbb{Z}_*^2} \tau_2^s\|m + n\tau\|^{-2s}$, for $\tau \in \mathbb{H}$, $\mathrm{Re}(s) > 1$ and otherwise by analytic continuation.
$\rho = \beta + i\gamma$	A complex zero of $\zeta(s)$; $\beta = \mathrm{Re}(\rho)$ and $\gamma = \mathrm{Im}(\rho)$.
$\rho_\chi = \beta + i\gamma$	A complex zero of $L(s, \chi)$; $\beta = \mathrm{Re}(\rho_\chi)$ and $\gamma = \mathrm{Im}(\rho_\chi)$.
$\eta(s)$	The Riemann η-function $\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.
$\xi(s)$	The Riemann ξ-function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$.
$\Xi(t)$	The Riemann Ξ-function $\xi(\frac{1}{2} + it)$.
$\rho(t)$	This is defined as $\eta(\frac{1}{2} + it)$.

d_F	The degree of an L-function F in $\mathcal{S}$.
q_F	The conductor of an L-function F in $\mathcal{S}$.
$H_F(n)$	The H-invariant of an L-function F in $\mathcal{S}$.
$\mathcal{B},\ \mathcal{B}_\chi$	In Chapter 1: these denote a certain way of bracketing ρ in sums of the type $\sum_\rho$ and $\sum_{\rho_\chi}$.
$N(T)$	The number of zeros ρ of $\zeta(s)$ for which $0 < \beta < 1$ and $0 < \gamma \le T$.
$N_0(T)$	The number of zeros ρ of $\zeta(s)$ for which $\beta = \frac{1}{2}$ and $0 < \gamma \le T$.
κ	Proportion of zeros on the critical line, i.e. $\kappa = \liminf_{T\to\infty} N_0(T)/N(T)$ as $T \to \infty$.
$\Delta(g)$	Laplace-Beltrami operator.
$\Gamma(s)$	The gamma function defined by $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $\mathrm{Re}(s) > 0$, and otherwise by analytic continuation.
$\mathrm{B}(x,y)$	The beta function defined by $\mathrm{B}(x,y) = \int_0^1 (1-u)^{x-1}u^{y-1}du$.
$\eta(\tau)$	The Dedekind eta-function defined by $\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^\infty (1 - e^{2\pi i n\tau})$ with $\tau \in \mathbb{H}$.
$J_\nu(x)$	This is the Bessel function of the first kind of order ν.
$K_\nu(x)$	This is the modified Bessel function of the second kind of order ν.
$Y_\nu(x)$	This is the Bessel function of the second kind of order ν.
${}_1F_1(a,b;x)$	This is the confluent hypergeometric function.
${}_2F_1(a,b,c;x)$	This is the Gauss hypergeometric function.
${}_pF_q$	Generalized hypergeometric function.
$\mathrm{Ci}(x)$	The cosine integral $\mathrm{Ci}(x) = -\int_x^\infty \cos t/tdt$.
$\mathrm{Si}(x)$	The sine integral $\mathrm{Si}(x) = \int_0^x \sin t/tdt$.
$\mathrm{si}(x)$	The complementary sine integral $\mathrm{si}(x) = -\int_x^\infty \sin t/tdt$.
$\mathrm{li}(x)$	This is logarithmic integral defined in Chapter 4.

$\mathrm{Ei}(x)$	This is exponential integral defined in Chapter 4.
$\mathcal{L}(x)$	This is $\pi^{-3/2}(e^x \operatorname{li}(e^{-x}) + e^{-x} \operatorname{li}(e^x))$.
$\hat{\vartheta}_3(x)$	The modified third elliptic function $\hat{\vartheta}_3(x) = \sum_{n=1}^{\infty} e^{-\pi x n^2}$.
$\varphi,\ \psi$	In Chapters 1 and 4: a pair of reciprocal functions under a certain kernel (cosine, Hankel or Koshliakov).
$\Theta(x)$	This is $\varphi(x) + \psi(x)$.
$S_{\mu,\nu}(z)$	This is the Lommel function.
$Z_1(s),\ Z_2(s)$	These denote the Mellin transforms of $\varphi(x)$ and $\psi(x)$, respectively, and normalized by a certain factor of $\Gamma(s/2)$.
$Z(s)$	This is $Z_1(s) + Z_2(s)$.
ψ	In Chapter 2: this denotes the saw-tooth function $\psi(t) := t - [t] - \frac{1}{2}$. In Chapter 4: this denotes the logarithmic derivative of Γ. In Chapter 5: this denotes a mollifier of $\zeta(s)$.
θ	In Chapter 4: this a positive parameter. In Chapter 5: it denotes the length of a mollifier.
$P[n]$	In Chapter 5: this is a short notation for $P(\log(y/n)/\log y)$ where P is a polynomial.
L	This is defined to be $L = \log T$ for large T.
σ_0	This is defined to be $\sigma_0 = 1/2 - R/L$, with R a bounded positive real number of our choice.
$\sum_{d\mid n}$	A sum taken over all positive divisors of n.
$\sum_{n\le x} f(n)$	A sum taken over all natural numbers not exceeding x; the empty sum being defined as zero.
$\prod_j$	A product taken over all possible values of the index j; the empty product defined to be unity.
$f(x) \sim g(x)$ as $x \to x_0$	This means that $\lim_{x\to x_0} f(x)/g(x) = 1$ with x_0 possibly infinite.
$f(x) = O(g(x))$	This means that $\lvert f(x)\rvert \le Cg(x)$ for $x \ge x_0$ and some absolute constant $C > 0$. Here $f(x)$ is a complex function of the real variable x.
$f(x) \ll g(x)$	This means the same as $f(x) = O(g(x))$.

$f(x) \gg g(x)$	This means the same as $g(x) = O(f(x))$.
$f(x) \asymp g(x)$	This means that both $f(x) \ll g(x)$ and $g(x) \gg f(x)$ hold.
$f(x) = o(g(x))$	This means that $\lim_{x \to x_0} f(x)/g(x) = 0$ with x_0 possibly infinite.
$\|x\|_\infty$	Infinity norm of x.
(x)	Fractional part of x.
$\lfloor x \rfloor, [x]$	Integer part of x.

Summary

While there is a unifying theme in the chapters that make up this thesis, it is very modular. The chapters are indeed fairly independent of each other.

The universal aspect behind them all is of course the Riemann zeta-function and its arithmetic counterparts, be it Möbius functions, Ramanujan sums, or more geometric objects such as tori and the special linear group $\mathrm{SL}(2, \mathbb{Z})$.

Each chapter contains a detailed introduction and motivation to study its associated topic. We summarize each chapter below.

I. A class of functions that satisfies intriguing explicit formulae of Ramanujan and Titchmarsh involving the zeros of an L-function in the reduced Selberg class of degree one and its associated Möbius function is studied. Moreover, a sufficient and necessary condition for the truth of the Riemann hypothesis due to Riesz is generalized. This is based on [KRR14].

II. The moments of the average of generalized Ramanujan sums are derived. Asymptotic results for an extension of a divisor problem and for an extension of a formula of Ramanujan are also provided. This chapter is taken from [RRa].

III. We consider a series of bounded vertical shifts of the Riemann ξ-function. Interestingly, although such functions have essential singularities, infinitely many of their zeros lie on the critical line. This material is from the first half of [DRRZ15].

IV. The second half of [DRRZ15] can be found here. We generalize some integral identities associated with the theta transformation formula and some formulae of G. H. Hardy and W. L. Ferrar in the context of a pair of functions reciprocal in Fourier cosine transform. The remaining generalizations in the context of a pair of functions reciprocal in Koshliakov transform will appear, along with other results, in [DRRZ].

V. We consider a 4-piece mollifier of the Riemann ζ-function which is made of the pieces introduced by Conrey and Levinson [Con89, Lev74], Bui, Conrey and Young [BCY11], Feng [Fen12] as well as a novel higher order piece. The associated mean value moment integrals of this mollifier are computed. A modest and temporary increment on the percentage of zeros of the Riemann zeta-function on the critical line is provided. These investigations will appear in [RRZ].

VI. A new and useful representation of zeta functions on complex tori is derived by using contour integration. It is shown to agree with the one obtained by using the Chowla-Selberg series formula, for which an alternative proof is thereby given. This is based on [EKRW15].

Additional work not covered in this thesis can be found in [KR15] and [RRb].

Chapter 1

On a class of functions that satisfies explicit formulae involving the Möbius function

1.1 Introduction and results

1.1.1 Motivation for studying the Möbius function

The Möbius function μ is defined as

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases} \tag{1.1}$$

If x denotes a positive real, then the Mertens function M is defined by

$$M(x) = \sum_{n \leqslant x} \mu(n).$$

The interest in studying $\mu(n)$ and $M(x)$ comes from their connection to the distribution of the prime numbers. For instance (see Hardy and Littlewood [HL18, §1.1]), the prime number theorem is equivalent to each of the following statements

$$M(x) = o(x) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0. \tag{1.2}$$

Estimates on Mertens's function date back to the 1880's when Mertens [Edw74] falsely conjectured that $M(x) \leqslant \sqrt{x}$ for all sufficiently large x. Later in 1885, Stieltjes [Edw74] claimed a proof of this conjecture. It was not until 100 years later that te Riele and Odlyzko [OtR85] disproved the Mertens' conjecture. Specifically they showed the following.

There are explicit constants $C_1 > 1$ and $C_2 < -1$ such that

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} \geqslant C_1, \quad \liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} \leqslant C_2.$$

This means that each of the inequalities $-\sqrt{x} \leq M(x)$ and $M(x) \leq \sqrt{x}$ fails for infinitely many x, or, equivalently, $M(x) = \Omega_{\pm}(\sqrt{x})$. The proof of te Riele and Odlyzko does not provide a specific value of x for which $M(x) \geq \sqrt{x}$. In [BT12] Best and Trudigan give an alternative disproof of Mertens' conjecture and they show that C_1 can be taken to be 1.6383 and C_2 to be -1.6383. The best unconditional estimate on the Mertens' function is (see Ivič [Ivi85, §12])

$$M(x) \ll x \exp(-c_1 \log^{\frac{3}{2}} x (\log\log x)^{-\frac{1}{5}}),$$

for $c_1 > 0$; and the bound on the assumption of the Riemann hypothesis is (see Titchmarsh [Tit86, §14.26])

$$M(x) \ll x^{\frac{1}{2}} \exp\left(\frac{c_2 \log x}{\log\log x}\right),$$

for $c_2 > 0$. The best unconditional Ω result for the Mertens function is

$$M(x) = \Omega_{\pm}(x^{\frac{1}{2}}),$$

and if $\zeta(s)$ has a zero of multiplicity m with $m > 1$ then

$$M(x) = \Omega_{\pm}(x^{\frac{1}{2}}(\log x)^{m-1}).$$

On the other hand, if the Riemann hypothesis is false, then

$$M(x) = \Omega_{\pm}(x^{\theta-\delta}),$$

where $\theta = \sup_{\rho,\zeta(\rho)=0} \mathrm{Re}(\rho)$ and δ is any positive constant (see Ingham [Ing64]).

1.1.2 Explicit formulae

An explicit formula is an equation which encapsulates certain arithmetical information and which involves the non-trivial zeros ρ of an L-function.

1.1.2.1 Ramanujan explicit formula

In 1918 Hardy and Littlewood (see [HL18, §2.5] and [Tit86, §9.8]) published an explicit formula suggested to them by Ramanujan. Recall that two functions $f(x)$ and $g(x)$ are cosine reciprocal if

$$\frac{\sqrt{\pi}}{2} f(x) = \int_0^\infty g(u)\cos(2ux)du \quad \text{and} \quad \frac{\sqrt{\pi}}{2} g(x) = \int_0^\infty f(u)\cos(2ux)du.$$

Under the benign assumption that the non-trivial zeros ρ are all simple, their explicit formula can be stated as follows.

Let a and b be two positive real numbers such that $ab = \pi$. Let φ and ψ be a pair of suitable cosine reciprocal functions. Let $Z_1(s)$ and $Z_2(s)$ be the Mellin transforms of $\varphi(x)$ and $\psi(x)$ respectively. Then

$$\begin{aligned}\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\varphi\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi\left(\frac{b}{n}\right) &= \frac{1}{\sqrt{a}}\sum_{\rho} a^{\rho}\frac{Z_1(1-\rho)}{\zeta'(\rho)} \\ &= -\frac{1}{\sqrt{b}}\sum_{\rho} b^{\rho}\frac{Z_2(1-\rho)}{\zeta'(\rho)}, \end{aligned} \tag{1.3}$$

provided the series involving ρ are convergent.

If we take $\varphi(x) = \psi(x) = \exp(-x^2)$, then it is easily seen that these functions are cosine reciprocal functions and that

$$Z_1(s) = Z_2(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right).$$

In this case (1.3) becomes

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a^2/n^2} - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-b^2/n^2} = \frac{1}{2\sqrt{a}}\sum_{\rho} a^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)} = -\frac{1}{2\sqrt{b}}\sum_{\rho} b^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}, \tag{1.4}$$

provided, once again, that the series

$$\sum_{\rho}\alpha^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}$$

is convergent for $\alpha > 0$. Hardy and Littlewood credit Ramanujan for first providing (1.4) and later on for suggesting the generalization (1.3). They did not, however, state the conditions that φ and ψ must satisfy for (1.3) to hold. The arithmetical information is contained in the Möbius function on the left-hand side of (1.3) and (1.4) and the analytic information is encoded in the sums involving the non-trivial zeros on either of the right-hand sides.

In 2013 Dixit [Dix13a] gave a one-variable generalization of (1.4). He showed the following result.

If we let a and b be positive reals such that $ab = 1$ and $z \in \mathbb{C}$, then

$$\begin{aligned}&\sqrt{a}e^{\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi a^2/n^2}\cos\left(\frac{\sqrt{\pi}az}{n}\right) - \sqrt{b}e^{-\frac{z^2}{8}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-\pi b^2/n^2}\cosh\left(\frac{\sqrt{\pi}bz}{n}\right) \\ &\qquad = -\frac{e^{-\frac{z^2}{8}}}{2\sqrt{\pi b}}\sum_{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}{}_1F_1\left(\frac{1-\rho}{2};\frac{1}{2};\frac{z^2}{4}\right)\pi^{\rho/2}b^{\rho}\end{aligned} \tag{1.5}$$

provided the series involving ρ are convergent, and where ${}_1F_1$ denotes the confluent hypergeometric function.

Clearly, if $z = 0$ then (1.5) becomes (1.4).

In [Dix12], Dixit obtained a character analogue of (1.4). To state his result we recall the following notation of the theory of Dirichlet L-functions. Suppose that χ is a character mod q. The indicator $\mathfrak{h}$ is defined by

$$\mathfrak{h} = \begin{cases} 0 & \text{if } \chi \text{ is even,} \\ 1 & \text{if } \chi \text{ is odd.} \end{cases} \tag{1.6}$$

The Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e^{2\pi i m/q}.$$

With this in mind, Dixit's second result is as follows.

Let a and b be two positive reals such that $ab = \pi$ and let χ denote a primitve Dirichlet character mod q *such that $\chi(-1) = (-1)^{\mathfrak{h}}$. If the non-trivial zeros ρ of $L(s,\chi)$ are all simple then one has*

$$\begin{aligned} &a^{\mathfrak{h}+1/2}\sqrt{\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^{1+\mathfrak{h}}} e^{-a^2/(qn^2)} - b^{\mathfrak{h}+1/2}\sqrt{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^{1+\mathfrak{h}}} e^{-b^2/(qn^2)} \\ &= q\frac{\sqrt{\tau(\chi)}}{2\sqrt{a}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{\Gamma(\frac{1+\mathfrak{h}-\rho}{2})}{L'(\rho,\chi)} = -q\frac{\sqrt{\tau(\bar{\chi})}}{2\sqrt{b}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{\Gamma(\frac{1+\mathfrak{h}-\rho}{2})}{L'(\rho,\bar{\chi})} \end{aligned} \tag{1.7}$$

provided the series involving ρ are convergent.

Later in [DRZ15b] Dixit, Roy and Zaharescu found the character analogue of (1.5) and in [DRZ15a] a generalization of (1.5) to Hecke forms.

The transformations in (1.3), (1.4), (1.5) and (1.7) exhibit a transformation of the type $x \to 1/x$, which is an analogue of the Poisson summation formula. These kinds of transformation formulas have broad interest in different branches of mathematics. In this article we establish a class of reciprocal functions, as well as a class of arithmetical functions obtained from a reduced Selberg class, which satisfies the transformation formula mentioned above. In [KRR14] a substantial number of examples are provided where the above transformations yield interesting special cases.

Let us suppose that $A_1 > 0$ and $T > 0$. We define the bracketing condition $\mathcal{B}$ on a sum involving the zeros $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ of $\zeta(s)$ to be a summation where the terms are bracketed in such a way that two terms for which

$$|\gamma - \gamma'| < \exp(-A_1|\gamma|/\log|\gamma|) + \exp(-A_1|\gamma'|/\log|\gamma'|) \tag{1.8}$$

are included in the same bracket. When a sum over ρ satisfies the bracketing condition $\mathcal{B}$ we will write $\sum_{\rho\in\mathcal{B}} f(\rho)$.

We define the bracketing condition $\mathcal{B}_\chi$ on a sum involving the zeros $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ of $L(s,\chi)$ to be a summation where the terms are bracketed in such a way that two terms for which

$$|\gamma - \gamma'| < \exp(-A_1|\gamma|/\log|\gamma| + 3) + \exp(-A_1|\gamma'|/\log|\gamma'| + 3) \tag{1.9}$$

are included in the same bracket. Similarly, when a sum over ρ satisfies the bracketing condition $\mathcal{B}_\chi$ we will write $\sum_{\rho\in\mathcal{B}_\chi} f(\rho)$.

If we assume that the zeros of $\zeta(s)$ satisfy the bracketing condition $\mathcal{B}$ then one can drop the assumption of convergence of the series in the right hand sides of (1.3), (1.4) and (1.5). Likewise, if we assume that the zeros of $L(s,\chi)$ satisfy the bracketing condition $\mathcal{B}_\chi$, then we can drop the assumption of convergence in the right-hand side of (1.7).

The size and the distribution of such bracketings are unknown but their existence is widely accepted. In fact, it is expected that the pairs of zeros $\{\rho,\rho'\}$ that need to be bracketed together in Ramanujan's explicit formula will occur rarely. For results on the correlation of zeros of L-functions, the reader is referred to Montgomery [Mon73], Rudnick and Sarnak [RS96], Katz and Sarnak [KS99b], [KS99a], Murty and Perelli [MP99], and Murty and Zaharescu [MZ02].

1.1.2.2 Titchmarsh explicit formula

An explicit formula for the Mertens function was first published in 1951 by Titchmarsh on the assumption of the Riemann hypothesis (see [Tit86, §14.27]), i.e. let $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. Specifically,

On RH and the simplicity of the non-trivial zeros, there exists a sequence T_ν, $\nu \le T_\nu \le \nu + 1$, such that

$$M(x) = -2 + \lim_{\nu\to\infty} \sum_{|\gamma|<T_\nu} \frac{x^\rho}{\rho\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!n\zeta(2n+1)} \tag{1.10}$$

if x is not an integer. If x is an integer, $M(x)$ is to be replaced by

$$M(x) - \tfrac{1}{2}\mu(x).$$

Note that, unlike RH, the assumption that the zeros are all simple is made for convenience. Indeed, this condition can be relaxed, and zeros with higher multiplicity can be accommodated at the cost of making the explicit formula much more complicated. Since it is widely believed that all zeros of the Riemann zeta-function are simple we shall operate under this assumption throughout.
In 1991 Bartz (see [Bar91a] and [Bar91b]) proved (1.10) unconditionally. A generalization to Cohen-Ramanujan sums of Bartz's results is established in [KR15] by the author and Kühn.

1.1.2.3 Weil explicit formula

The von Mangoldt function is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some } m \in \mathbb{N} \text{ and prime } p, \\ 0, & \text{otheriwse.} \end{cases}$$

In 1952 Weil (see [IK04, §5.5] and [Wei52]) published an explicit formula for the von Mangoldt function.

Suppose that f is $\mathcal{C}^\infty$ and compactly supported. Moreover, denote by F its Mellin transform. Then

$$\sum_\rho F(\rho) + \sum_{n=1}^{\infty} F(-2n) = F(1) + \sum_{n=1}^{\infty} \Lambda(n) f(n). \tag{1.11}$$

In order to state the main theorems proved in this chapter, we first need to introduce some further concepts.

1.1.3 Hankel transformations

Two functions $\varphi(x)$ and $\psi(x)$ are said to be reciprocal under the Hankel transformation of order ν if

$$\varphi(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux)\psi(u)du \quad \text{and} \quad \psi(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)du, \tag{1.12}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν defined as the solution to (A.8). It can be written as

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

The existence of such reciprocity was first shown by Titchmarsh (see [Tit22] and [Tit48]). In particular he showed the following.

If $\varphi(s)$ is integrable in the sense of Lebesgue and $\nu \geq -\frac{1}{2}$ then

$$\int_0^a (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)du$$

converges in mean to a function $\psi(x)$ of integrable square in $(0,\infty)$ as $a \to \infty$.

Hankel transformations reduce to Fourier's cosine and sine transforms for $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$, respectively. The Mellin transforms of $\varphi(x)$ and $\psi(x)$ are defined, as usual, by

$$Z_1(s) = \int_0^\infty x^{s-1}\varphi(x)dx \quad \text{and} \quad Z_2(s) = \int_0^\infty x^{s-1}\psi(x)dx. \tag{1.13}$$

As explained in (A.27), their inverse Mellin transforms are given by

$$\varphi(x) = \frac{1}{2\pi i}\int_{(c)} Z_1(s)x^{-s}ds \quad \text{and} \quad \psi(x) = \frac{1}{2\pi i}\int_{(c)} Z_2(s)x^{-s}ds. \tag{1.14}$$

The value of c will depend on the nature of the functions φ and ψ.

Definition 1.1. Let $0 < \omega \leq \pi$ and $\alpha < \frac{1}{2}$. If $f(z)$ is such that

i) $f(z)$ is analytic with $z = re^{i\theta}$, regular in the angle defined by $r > 0$, $|\theta| < \omega$,

ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O(|z|^{-\alpha-\epsilon}), & \text{if } |z| \text{ is small,} \\ O(e^{-|z|}), & \text{if } |z| \text{ is large,} \end{cases} \tag{1.15}$$

for every positive ϵ and uniformly in any angle $|\theta| < \omega$,

then we say that f belongs to the class K and write $f(z) \in K(\omega, \alpha)$.

1.1.4 The Selberg class

In [Sel92], Selberg introduced a general class $\mathcal{S}$ of L-functions. Let F be an L-function in $\mathcal{S}$, then the completed L-function is defined by

$$\Lambda(s) = Q^s \prod_{i=1}^{d} \Gamma(\alpha_i s + r_i) F(s) \tag{1.16}$$

where $Q > 0$, $\alpha_i > 0$, $r_i \in \mathbb{C}$ with $\mathrm{Re}(r_i) \geq 0$. The degree d_F and conductor q_F are defined by

$$d_F = 2 \sum_{j=1}^{d} \alpha_j \quad \text{and} \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{d} \alpha_j^{2\alpha_j}, \tag{1.17}$$

respectively. It is conjectured that the degree d_F and conductor q_F are both integers. For a non-negative integer n, the H-invariants are defined by

$$H_F(n) = 2 \sum_{j=1}^{d} \frac{B_n(r_j)}{\alpha_j^{n-1}},$$

where $B_n(x)$ are the familiar n-th Bernoulli polynomials. The first few $B_n(x)$'s are given by

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad \cdots .$$

Hence we find that

$$H_F(0) = d_F, \quad H_F(1) = 2 \sum (r_j - 1/2), \quad \cdots . \tag{1.18}$$

1.1.5 Main results

Equipped with these notions our first result is as follows.

Theorem 1.2. *Suppose that F is an element of the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$ and $H_F(1) = -\nu - \frac{1}{2}$. Let $\frac{\pi}{4} < \omega \leq \pi$, $\alpha < \frac{1}{2}$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal functions under the Hankel transformation of order ν. Let $Z_1(s)$ and $Z_2(s)$ be defined as above and let x be a positive real. Then there exists a sequence $\{T_l\}$ of positive numbers that satisfies the following.*

i) If $q_F = 1$ then

$$\sum_{n=1}^{\infty}\mu(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty}\sum_{-T_l<\mathrm{Im}(\rho)<T_l}\frac{Z_1(\rho)}{\zeta'(\rho)}x^{\rho} + \sqrt{2\pi}\sum_{k=1}^{\infty}\frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)}\left(\frac{x}{2\pi}\right)^{-k}. \tag{1.19}$$

ii) If $q := q_F \geq 2$ then there exists a primitive Dirichlet character χ mod q with $\chi(-1) = -2\nu$ such that

$$\sum_{n=1}^{\infty}\mu(n)\chi(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty}\sum_{-T_l<\mathrm{Im}(\rho)<T_l}\frac{Z_1(\rho)}{L'(\rho,\chi)}x^{\rho} + \frac{Z_1(s_0)}{L'(s_0,\chi)}x^{s_0}$$
$$+ i^{\frac{1}{2}+\nu}\frac{\sqrt{2\pi}}{\tau(\chi)}\sum_{k=0}^{\infty}\frac{(-1)^k Z_2(1+k)}{(k!)^2 L(1+k,\bar{\chi})}\left(\frac{qx}{2\pi}\right)^{-k}$$

on the assumption that the Riemann hypothesis for Dirichlet L-functions is true and where s_0 denotes a hypothetical Landau-Siegel zero.

Equation (1.19) is reminiscent of the Weil explicit formula except that $\Lambda(n)$ is replaced by $\mu(n)$. Similar formulae due to Berndt [Ber71] and Ferrar (see [Fer35], [Fer37], and [Tit48, §2.9]) for the divisor function $d(n)$ exist as well. Extensions of the Weil explicit formula (1.11) to generalized von Mangoldt functions and other arithmetical functions such as the Liouville λ function can be found in another article by the author and Roy [RRb]. The second result is as follows.

Theorem 1.3. *Suppose that F is an element of the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$ and $H_F(1) = \nu - \frac{1}{2}$. Let $\frac{\pi}{4} < \omega \leq \pi$, $\alpha < \frac{1}{2}$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal functions under the Hankel transformation of order ν. Let $Z_1(s)$ and $Z_2(s)$ defined as above. If a and b are two positive reals such that $ab = 2\pi$, then one has the following.*

i) If $q_F = 1$ then

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\varphi\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi\left(\frac{b}{n}\right) = \frac{1}{\sqrt{a}}\sum_{\rho\in\mathcal{B}}a^{\rho}\frac{Z_1(1-\rho)}{\zeta'(\rho)}$$
$$= -\frac{1}{\sqrt{b}}\sum_{\rho\in\mathcal{B}}b^{\rho}\frac{Z_2(1-\rho)}{\zeta'(\rho)}. \tag{1.20}$$

ii) If $q := q_F \geq 2$ then there exists a primitive Dirichlet character χ mod q with $\chi(-1) = -2\nu$ such that

$$\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right) \tag{1.21}$$
$$= \frac{q^{1/2}\sqrt{\tau(\chi)}}{a^{1/2}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{a}{q^{1/2}}\right)^{\rho}\frac{Z_1(1-\rho)}{L'(\rho,\chi)} = -\frac{q^{1/2}\sqrt{\tau(\bar{\chi})}}{b^{1/2}}\sum_{\rho\in\mathcal{B}_\chi}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.$$

If one changes (1.15) to the following

$$f(z) = \begin{cases} O(|z|^{-\alpha-\epsilon}), & \text{if } |z| \text{ is small,} \\ O(|z|^{-\beta-\epsilon}), & \text{if } |z| \text{ is large,} \end{cases} \tag{1.22}$$

with $\alpha = 0$ and $\beta > 1$, then Theorem 1.3 would also hold for φ and ψ satisfying the above growth conditions.

One can see the condition $H_F(1) = \nu - \frac{1}{2}$ is necessary. This condition naturally leads us to make the following conjecture.

Conjecture 1.4. *Let F be an element in the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$, $\frac{\pi}{2} < \omega \leq \pi$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal under the Hankel transformation of order ν. Then* (1.20) *holds only when $\nu = -1/2$ and* (1.21) *holds only when $\nu = \pm 1/2$.*

Remark 1.5. The following special cases are to be noted.

1. Let $\varphi(x) = \psi(x) = x^{(\nu+1/2)} e^{-\frac{x^2}{2}}$ for $\nu = \pm 1/2$. Clearly $\varphi, \psi \in K(\omega, a)$. Also

$$Z_1(s) = Z_2(s) = \left(\frac{1}{2}\right)^{(\frac{\nu}{2}-\frac{3}{4})} 2^{\frac{s}{2}} \Gamma\left(\frac{s+\nu+1/2}{2}\right).$$

   If we substitute the above values of φ, ψ, Z_1 and Z_2 in (1.21) then we obtain (1.7).

2. Let $\varphi(x) = e^{-x^2-z^2/2}\cosh(zx)$ and $\psi(x) = e^{-x^2+z^2/2}\cos(zx)$. One can see that $\varphi, \psi \in K(\omega, a)$ and that they are reciprocal under cosine transformations, i.e. $\nu = -1/2$. Their Mellin transformations are given by

$$Z_1(s) = \frac{1}{2} e^{-\frac{z^2}{8}} \Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{s}{2}, \frac{1}{2}; \frac{z^2}{4}\right),$$
$$Z_2(s) = \frac{1}{2} e^{\frac{z^2}{8}} \Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{s}{2}, \frac{1}{2}; -\frac{z^2}{4}\right).$$

   If we substitute the above values of φ, ψ, Z_1 and Z_2 in (1.20) and (1.21) then we obtain (1.5) and [DRZ15b, Theorem 1.2, part i)] respectively.

3. Let $\varphi(x) = e^{-x^2-z^2/2}\sinh(zx)$ and $\psi(x) = e^{-x^2+z^2/2}\sin(zx)$. One can see that $\varphi, \psi \in K(\omega, a)$ and that they are reciprocal under sine transformations, i.e. $\nu = 1/2$. Their Mellin transformations are given by

$$Z_1(s) = \frac{z}{2} e^{-\frac{z^2}{8}} \Gamma\left(\frac{1+s}{2}\right) {}_1F_1\left(\frac{1+s}{2}, \frac{3}{2}; \frac{z^2}{4}\right),$$
$$Z_2(s) = \frac{z}{2} e^{\frac{z^2}{8}} \Gamma\left(\frac{1+s}{2}\right) {}_1F_1\left(\frac{1+s}{2}, \frac{3}{2}; -\frac{z^2}{4}\right).$$

   If we substitute the above values of φ, ψ, Z_1 and Z_2 in (1.21) then we obtain [DRZ15b, Theorem 1.2, part ii)].

Finally, on inspiration coming from (1.4) Hardy and Littlewood [HL18] found an equivalent condition for the validity of the Riemann hypothesis. This kind of result was first published by Riesz in [Rie16]. The analogues of the conditions for the Dirichlet L-functions and Hecke forms were obtained in [DRZ15b] and [DRZ15a] respectively. The motivation for the following theorem comes from the following heuristics. Let us suppose that φ and ψ meet the conditions of the previous theorems and that $d_F = q_F = 1$. For $y > 0$, let us define the functions

$$P_\varphi(y) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \varphi\left(\frac{y}{n}\right) \quad \text{and} \quad P_\psi(y) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \psi\left(\frac{y}{n}\right).$$

Now we perform a Maclaurin expansion of φ around $y = 0$ to obtain

$$P_\varphi(y) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=0}^{\infty} \left(\frac{y}{n}\right)^k \frac{\varphi^{(k)}(0)}{k!} = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} y^k \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{k+1}} = \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \frac{y^k}{\zeta(1+k)},$$

with a similar formula holding for $P_\psi(y)$. The interchange is justified by the fact that φ is in $K(\omega, \alpha)$ so that φ can be written as a convergent Taylor series at 0. The explicit formula (1.20) can be written as

$$a^{\frac{1}{2}} P_\varphi(a) - b^{\frac{1}{2}} P_\psi(b) = -\sum_\rho b^{\rho - \frac{1}{2}} \frac{Z_2(1-\rho)}{\zeta'(\rho)}. \tag{1.23}$$

If we assume the Riemann hypothesis, $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$, and the absolute convergence of

$$\sum_\rho b^{i\gamma} \frac{Z_2(1-\rho)}{\zeta'(\rho)}, \tag{1.24}$$

then the right-hand side of (1.23) is of the form $O(1)$ when $b \to \infty$. Thus the left hand side of (1.23) is now $-b^{\frac{1}{2}} P_\psi(b) \ll 1$, or, equivalently

$$\sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{k!} \frac{b^k}{\zeta(1+k)} \ll b^{-\frac{1}{2}}, \tag{1.25}$$

as $b \to \infty$. Seeing how the Riemann hypothesis and the convergence of (1.24) implies the bound (1.25), we will now establish the following theorem which provides an equivalence of the Riemann hypothesis.

Theorem 1.6. *Let us suppose that φ is in $K(\omega, 0)$ and that it is analytic at 0. Consider the function*

$$P_\varphi(y) := \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \frac{y^k}{\zeta(1+k)}.$$

One has the following:

i. *The Riemann hypothesis implies $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$.*

ii. *If $Z_1(-s)$ has no zeros in the interval $-\frac{1}{2} < \mathrm{Re}(s) \le 0$, then the estimate $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$ implies the Riemann hypothesis.*

Remark 1.7. If $Z_1(-s)$ had zeros then all the zeros of $\zeta(s)$ would still lie on the critical line except for the zeros that coincide with the zeros of $Z_1(-s)$. In most of the examples that we considered in [KRR14] we see that $Z_1(-s)$ has at most finitely many zeros in the region $-\frac{1}{2} < \mathrm{Re}(s) \le 0$.

1.2 Preliminary Lemmas

We will use the following lemmas to prove our main theorems.

Lemma 1.8. *Let $\varphi, \psi \in L^2(0,\infty)$ be two reciprocal Hankel transforms of order ν. Then*

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}-it}dt, \tag{1.26}$$

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}-it}dt, \tag{1.27}$$

the integrals are mean square integrable, $2^{\frac{it}{2}}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})\Phi(\frac{1}{2}+it)$ and $2^{\frac{it}{2}}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})\Psi(\frac{1}{2}+it)$ belong to $L^2(-\infty,\infty)$, and

$$\Phi\left(\frac{1}{2}-it\right) = \Psi\left(\frac{1}{2}+it\right). \tag{1.28}$$

Proof. Suppose that φ belongs to $L^2(0,\infty)$. One can see that

$$\int_0^{\infty}\varphi^2(x)dx = \int_{-\infty}^{\infty}\varphi^2(e^x)e^x dx.$$

Hence $F(x) := \varphi(e^x)e^{x/2} \in L^2(-\infty,\infty)$. Then from the theory of Fourier transforms (see [Tit48]) it follows that

$$Z_1\left(\frac{1}{2}+it\right) = \int_{-\infty}^{\infty}F(x)e^{itx}dx = \int_0^{\infty}\varphi(x)x^{-\frac{1}{2}+it}dx \tag{1.29}$$

exists as a mean square integral for almost all t. Also $Z_1(\frac{1}{2}+it) \in L^2(-\infty,\infty)$ and

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}Z_1\left(\frac{1}{2}+it\right)e^{-ixt}dt. \tag{1.30}$$

The above integral is also a mean square integral. In other words, (1.30) can be written as

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}Z_1\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}+it}dt. \tag{1.31}$$

Similarly we obtain

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}Z_2\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}+it}dt. \tag{1.32}$$

Let us consider two functions Φ and Ψ such that

$$Z_1\left(\frac{1}{2}+it\right) = 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}+it\right) \tag{1.33}$$

and

$$Z_2\left(\frac{1}{2}+it\right) = 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right). \tag{1.34}$$

Replacing the above equalities in (1.31) and (1.32) we obtain (1.26) and (1.27). Now we complete the proof by proving (1.28). For all $\nu \geq -1/2, y > 0$ and $x > 0$ we have

$$\int_0^y \sqrt{ux}J_\nu(ux)\,du = \frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left(\begin{matrix}\frac{\nu}{2}+\frac{3}{4}\\ \frac{\nu}{2}+\frac{7}{4},\nu+1\end{matrix}; -\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}. \tag{1.35}$$

The right-hand side of (1.35) belongs to $L^2(0,\infty)$ and the Mellin transform is given by

$$\int_0^\infty \frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4},\nu+1};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}x^{-\frac{1}{2}+it}dx = \frac{2^{it}y^{\frac{1}{2}-it}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})}{(\frac{1}{2}-it)\Gamma(\frac{\nu}{2}+\frac{1}{2}-\frac{it}{2})}. \tag{1.36}$$

This follows by taking equation (10.22.10) of [OM14] followed by equation (10.1) of [Obe74]. We also have that $\varphi \in L^2(0,\infty)$ and its Mellin transform is given by (1.29). Hence by an analogue of Plancherel's theorem for Mellin transform (see [Tit48, Theorem 72]) we have

$$\begin{aligned} &\int_0^\infty \varphi(x)\frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4},\nu+1};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{2^{it}y^{\frac{1}{2}-it}\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2})}{(\frac{1}{2}-it)\Gamma(\frac{\nu}{2}+\frac{1}{2}-\frac{it}{2})}Z_1\left(\frac{1}{2}-it\right)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}-it\right)\frac{y^{\frac{1}{2}-it}}{\frac{1}{2}-it}dt, \end{aligned} \tag{1.37}$$

where in the ultimate step we have used (1.33). Now from (1.12) we have

$$\psi(u) = \lim_{a\to\infty}\int_0^a \sqrt{ux}J_\nu(ux)\varphi(x)dx,$$

where the limit converges in the sense of mean-square. Therefore for all $x>0, y>0$ and $\nu \geq -1/2$ we find that

$$\begin{aligned} \int_0^y \psi(u)\,du &= \lim_{a\to\infty}\int_0^y\int_0^a \sqrt{ux}J_\nu(ux)\varphi(x)dx\,du \\ &= \lim_{a\to\infty}\int_0^a \varphi(x)\frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4},\nu+1};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}dx \\ &= \int_0^\infty \varphi(x)\frac{y(xy)^{\nu+\frac{1}{2}}{}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4},\nu+1};-\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}dx. \end{aligned} \tag{1.38}$$

The left-hand side of (1.38) is

$$\begin{aligned} \int_0^y \psi(u)\,du &= \frac{1}{2\pi}\int_0^y\int_{-\infty}^\infty 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right)u^{-\frac{1}{2}-it}dt\,du \\ &= \frac{1}{2\pi}\left(\lim_{X\to\infty}\int_0^y\int_0^X + \lim_{Y\to\infty}\int_0^y\int_{-Y}^0\right)2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right)\frac{dtdu}{u^{\frac{1}{2}+it}} \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right)\frac{y^{\frac{1}{2}-it}}{\frac{1}{2}-it}dt. \end{aligned} \tag{1.39}$$

By (1.38) we see that the right-hand sides of (1.37) and (1.39) are equal. Hence from [Tit48, Theorem 32] we conclude that

$$\Phi\left(\frac{1}{2}-it\right) = \Psi\left(\frac{1}{2}+it\right),$$

and this ends the proof. □

Lemma 1.9. *Let φ and ψ be reciprocal functions under the Hankel transformation of order ν defined in (1.12). Let $\varphi, \psi \in K(\omega, \alpha)$. Then there exist two regular functions Φ and Ψ such that*

$$\varphi(x) = \frac{1}{2\pi i}\int_{(c)} 2^{\frac{s}{2}-\frac{1}{4}}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)\Phi(s)x^{-s}ds, \tag{1.40}$$

$$\psi(x) = \frac{1}{2\pi i}\int_{(c)} 2^{\frac{s}{2}-\frac{1}{4}}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)\Psi(s)x^{-s}ds \tag{1.41}$$

for $c > 0$. Moreover Φ and Ψ satisfy the following:

1 $\Phi(s) = \Psi(1-s)$ *for all* $s \in \mathbb{C}$,

2 $\Psi(s) \ll e^{(\frac{\pi}{4}-\omega+\epsilon)|t|}$ *for every positive ϵ and uniformly for $\sigma \in \mathbb{R}$.*

Remark 1.10. If φ and ψ satisfy (1.22), then conditions (1) and (2) in Lemma 1.9 hold uniformly for $\alpha < \sigma < \beta$.

Proof of Lemma 1.9. Since $\varphi, \psi \in K(\omega, \alpha)$, the right hand sides of (1.13) are absolutely convergent. Then it follows that $Z_1(s)$ and $Z_2(s)$ are regular in $\alpha < \sigma$. Let

$$Z_1(s) = 2^{\frac{s}{2}-\frac{1}{4}}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)\Phi(s), \tag{1.42}$$

and

$$Z_2(s) = 2^{\frac{s}{2}-\frac{1}{4}}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)\Psi(s). \tag{1.43}$$

Hence by (1.33) and (1.34) of Lemma 1.8, we deduce that $\Phi(s)$ and $\Psi(s)$ also regular in this region. One can see $\varphi, \psi \in L^2$. Therefore from (1.28) of Lemma 1.8, $\Psi(s) = \Phi(1-s)$ for $\sigma = 1/2$. Thus, by analytic continuation $\Psi(s) = \Phi(1-s)$ for $\alpha < \sigma$ and hence for all $s \in \mathbb{C}$. Also (1.40) and (1.41) hold for $\alpha < c = \sigma$. Let us consider the line along any radius vector r and angle θ, where $|\theta| < \omega$. Then by Cauchy's theorem we can deform the integral (1.13) to

$$Z_1(\sigma+it) = \int_0^\infty r^{\sigma+it-1}e^{i\theta(\sigma+it)}\varphi(re^{i\theta})\,dr,$$

where $\theta, t > 0$. Therefore

$$|Z_1(\sigma+it)| = \left|\int_0^\infty r^{\sigma+it-1}e^{i\theta(\sigma+it-1)}\varphi(re^{i\theta})\,dr\right| \le e^{-\theta t}\int_0^\infty r^{\sigma-1}|\varphi(re^{i\theta})|dr \ll e^{-|\theta||t|}, \tag{1.44}$$

since $\varphi \in K(\omega, \alpha)$ as $|t| \to \infty$. Now combining (1.42), (1.44) and (A.6) we get

$$\Psi(1-s) = \Phi(s) \ll e^{(\frac{\pi}{4}-|\theta|)|t|} \ll e^{(\frac{\pi}{4}-\omega+\eta)|t|}, \tag{1.45}$$

for every positive η. This proves the lemma. □

Remark 1.11. For functions which are self-reciprocal under the Hankel transformation, similar results to Lemmas 1.8 and 1.9 were obtained in [HT31] in a vertical strip.

The following result is Theorem 3 from Kaczorowski and Perelli [KP99].

Lemma 1.12. *Let $F \in \mathcal{S}$. Suppose that $d_F = 1$ and $\mathrm{Re}(H_F(1))$ is either 0 or 1. If $q_F = 1$ then $F(s) = \zeta(s)$. If $q_F \geq 2$ then there exists a primitive Dirichlet character χ mod q_F with $\chi(-1) = -(2\,\mathrm{Re}(H_F(1)) + 1)$ such that $F(s) = L(s + i\,\mathrm{Im}(H_F(1)), \chi)$.*

Remark 1.13. It is worthwhile to mention pertinent observations which motivated the authors to study the case $d_F = 1$. The following results are due to Conrey and Ghosh [CG93] and Kaczorowski and Perelli [KP99, KP02, KP11].

1. One has $d_F = 0$ precisely when $F = 1$.
2. There is no function $F \in S$ with $0 < d_F < 1$.
3. There is no function $F \in S$ with $1 < d_F < 2$.

The following results due to Montgomery [Mon77]; Ramachandra and Balasubramanian [Ram74], [Ram77] and [BR77] will enable us to prove Theorem 1.1 with $d_F = q_F = 1$ without the assumption of the Riemann hypothesis.

Lemma 1.14. *For any given $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon)$ such that $T \geq T_0$ and the following holds: between T and $2T$ there exists a real number t for which*

$$|\zeta(\sigma \pm it)|^{-1} < c_1 t^{\varepsilon}$$

for $-1 \leq \sigma \leq 2$ with an absolute constant $c_1 > 0$.

For the case where $q_F > 1$, the analogue results are due to Soundararajan, [Sou08]; Lamzouri, [Lam11]. However, this latter one depends on the truth of the Riemann hypothesis for Dirichlet L-functions.

Lemma 1.15. *Assume the Riemann hypothesis for Dirichlet L-functions. For any given $\varepsilon > 0$ and primitive Dirichlet character χ mod q there exists a $T_0 = T_0(\varepsilon, q)$ such that if $T \geq T_0$ then the following holds: between T and $2T$ there exists a real number t for which*

$$|L(\sigma \pm it, \chi)|^{-1} < c(q) t^{\varepsilon}$$

for $-1 \leq \sigma \leq 2$ with an absolute constant $c(q) > 0$.

An intermediate result we will be using is due to Ahlgren, Berndt, Yee and Zaharescu [ABYZ04].

Lemma 1.16. *If χ is a primitive character of conductor N and k an integer ≥ 2 such that $\chi(-1) = (-1)^k$ then one has*

$$\frac{(k-2)!N^{k-2}\tau(\chi)}{2^{k-1}\pi^{k-2}i^{k-2}} L(k-1, \bar{\chi}) = L'(2-k, \chi). \tag{1.46}$$

1.3 Proof of Theorem 1.2

i) Let F be a Selberg L-function of degree $d_F = 1$ and conductor $q_F = 1$. Then by Lemma 1.12 we see that $F(s) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function. Therefore there is only one gamma factor in the completed Selberg L-function of F for which $r_j = 0$ and $\lambda_j = 1/2$. From (1.18) we see that $H_F(1) = -1$ when $r_j = 0$ and hence $\nu = 1/2$. Therefore $\varphi, \psi \in K(\omega, \alpha)$ is a pair of reciprocal sine transformations. Now

$$\sum_{n=1}^{\infty} \mu(n)\varphi\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} \mu(n) \int_{(\lambda)} Z_1(s)\left(\frac{x}{n}\right)^s ds$$
$$= \frac{1}{2\pi i}\int_{(\lambda)} Z_1(s)x^s\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right)ds. \tag{1.47}$$

By Lemma 1.9 $Z_1(s) \ll e^{(-\omega+\eta)|t|}$ for every positive η. For $1 < \lambda < 2$ the sum inside the above integral is absolutely convergent. Hence, the far right-hand side of above equalities is absolutely convergent, which justifies the interchange of the summation and integration. Recall the Dirichlet series valid for $\mathrm{Re}(s) > 1$ of the Möbius function:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

From (1.33) we find that the simple poles of $Z_1(s)$ are at $s = -2k+1$ for $k = 0, 1, 2, \cdots$. For $1 < \lambda < 2$ and $-1 < c < 0$ we consider the positively oriented closed contour $\Omega = [c - iT, c + iT, \lambda + iT, \lambda - iT]$ where $T > 0$. Thus by the residue theorem

$$\frac{1}{2\pi i}\oint_{\Omega}\frac{Z_1(s)}{\zeta(s)}x^s ds = \sum_{-T<\mathrm{Im}(\rho)<T}\lim_{s\to\rho}(s-\rho)\frac{Z_1(s)}{\zeta(s)}x^s = \sum_{-T<\mathrm{Im}(\rho)<T}\frac{Z_1(\rho)}{\zeta'(\rho)}x^{\rho}. \tag{1.48}$$

The functional equation of $\zeta(s)$ is given by

$$\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s). \tag{1.49}$$

From Lemma 1.9 we have

$$Z_1(s) = 2^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma(1-\frac{s}{2})}Z_2(1-s). \tag{1.50}$$

Hence by using (1.49), (1.50) and the duplication of the gamma function we find

$$\int_{c-iT}^{c+iT}\frac{Z_1(s)}{\zeta(s)}x^s ds = \sqrt{2\pi}\int_{c-iT}^{c+iT}\left(\frac{x}{2\pi}\right)^s\frac{\Gamma(s)}{\Gamma(1-s)}\frac{Z_2(1-s)}{\zeta(1-s)}ds. \tag{1.51}$$

Now we consider the positive oriented contour Ω' with vertices $[-N-\frac{1}{2}-iT, c-iT]$, $[c-iT, c+iT]$, $[c+iT, -N-\frac{1}{2}+iT]$ and $[-N-\frac{1}{2}+iT, -N-\frac{1}{2}-iT]$. The poles of the integrand of the right-hand side integral of (1.51) are at $k = -1, -2, -3, \ldots$. By the residue theorem we have

$$\frac{\sqrt{2\pi}}{2\pi i}\oint_{\Omega'}\left(\frac{x}{2\pi}\right)^s\frac{\Gamma(s)}{\Gamma(1-s)}\frac{Z_2(1-s)}{\zeta(1-s)}ds = \sqrt{2\pi}\sum_{k=1}^{N}\frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)}\left(\frac{x}{2\pi}\right)^{-k}. \tag{1.52}$$

Therefore by Lemma 1.9 and equation (A.7) we have

$$\int_{-N-\frac{1}{2}-iT}^{-N-\frac{1}{2}+iT}\left(\frac{x}{2\pi}\right)^s\frac{\Gamma(s)}{\Gamma(1-s)}\frac{Z_2(1-s)}{\zeta(1-s)}ds$$

$$\ll \int_{-T}^{T} \left(\frac{x}{2\pi}\right)^{-N-\frac{1}{2}} \frac{e^{2(N+1)-2(N+1)\log(\sqrt{t^2+(N+1/2)^2})}}{e^{(\pi+\omega+\eta)|t|}} dt, \tag{1.53}$$

which tends to zero as $N \to \infty$ for any fixed T. Combining (1.52) and (1.53) we find

$$\begin{aligned}\sqrt{2\pi}\int_{c-iT}^{c+iT} \left(\frac{x}{2\pi}\right)^{s} \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds &= \sqrt{2\pi}\sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)} \left(\frac{x}{2\pi}\right)^{-k} \\ &+ \sqrt{2\pi}\left(\int_{-\infty-iT}^{c-iT} + \int_{-\infty+iT}^{c+iT}\right) \left(\frac{x}{2\pi}\right)^{s} \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds.\end{aligned} \tag{1.54}$$

Similarly as with (1.53) we have

$$\begin{aligned}\int_{-\infty\pm iT}^{c\pm iT} \left(\frac{x}{2\pi}\right)^{s} \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds &\ll \int_{-\infty}^{c} \left(\frac{x}{2\pi}\right)^{\sigma} \frac{e^{1-2\sigma+(2\sigma-1)\log(\sqrt{T^2+\sigma^2})}}{e^{(\pi+\omega+\eta)T}} d\sigma \\ &\ll \frac{1}{e^{(\pi+\omega+\eta)T}}.\end{aligned} \tag{1.55}$$

Now by Lemmas 1.9 and 1.14 we have

$$\int_{c\pm iT}^{\lambda\pm iT} \frac{Z_1(s)}{\zeta(s)} x^s ds \ll T^{\epsilon} e^{(-\omega+\eta)T}, \tag{1.56}$$

where T runs through a sequence $\{T_l\}$ with $T_l > T_0(\epsilon)$. Here ϵ and η are any positive numbers. Now combine (1.47), (1.48), (1.51), (1.52), (1.53) and (1.56) to conclude

$$\sum_{n=1}^{\infty} \mu(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty} \sum_{-T_l<\mathrm{Im}(\rho)<T_l} \frac{Z_1(\rho)}{\zeta'(\rho)} x^{\rho} + \sqrt{2\pi}\sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)} \left(\frac{x}{2\pi}\right)^{-k}.$$

This proves part i) of Theorem 1.1.

ii) In this case we consider that F is an L-function of degree $d_F = 1$ and conductor $q_F \geq 2$. Using Lemma 1.12 we find $F(s) = L(s,\chi)$ for some Dirichlet primitive character mod q_F. Therefore the completed L-function of F contains only one gamma factor and hence $r_j = 0$ or $r_j = 1/2$. Since ν is real then $\mathrm{Im}(H_F(1)) = 0$ and hence $H_F(1) = -1$ or $H_F(1) = 0$. By Lemma 1.9 we know that $\Phi(s)$ is analytic on the whole complex plane. Thus the poles of $Z_1(s)$ are at the poles of $\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)$. If $\nu = -1/2$ then $s = 0$ is a pole $Z_1(s)$. For the sake of brevity we will prove the case where χ is a even character mod q_F; that is, when $\nu = 1/2$. The other case is handled in a similar fashion. In this case $Z_1(s)$ is analytic for $\mathrm{Re}(s) > -1$. Arguing as in part i) we have

$$\sum_{n=1}^{\infty} \mu(n)\chi(n)\varphi\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{(\lambda)} \frac{Z_1(s)}{L(s,\chi)} x^s ds. \tag{1.57}$$

Consider the positively oriented contour Ω mentioned in part i). By the residue theorem one can find

$$\frac{1}{2\pi i}\oint_{\Omega} \frac{Z_1(s)}{L(s,\chi)} x^s ds = \frac{Z_1(0)}{L'(0,\chi)} + \sum_{-T<\mathrm{Im}(\rho)<T} \frac{Z_1(\rho)}{L'(\rho,\chi)} x^{\rho}, \tag{1.58}$$

where the ρ's denote the non-trivial zeros $L(s,\chi)$, assumed to be simple for notational convenience. If there is a Landau-Siegel zero (see §14 of [Dav00]) at $s = s_0$ then we would have to add the extra term

$$\operatorname*{res}_{s=0} \frac{Z_1(s)}{L(s,\chi)} x^s = \frac{Z_1(s_0)}{L'(s_0,\chi)} x^{s_0}.$$

We note that this hypothetical zero is real and simple. Using the functional equation of Lemma 1.9 and the relation in Lemma 1.16 we find that

$$\frac{Z_1(0)}{L'(0,\chi)} = \frac{\sqrt{2\pi}}{\tau(\chi)} \frac{Z_2(1)}{L(1,\bar{\chi})}. \tag{1.59}$$

Proceeding as in the proof of part i) we have

$$\begin{aligned}\int_{c-iT}^{c+iT} \frac{Z_1(s)}{L(s,\chi)} x^s ds &= \frac{\sqrt{2\pi}}{\tau(\chi)} \int_{c-iT}^{c+iT} \left(\frac{qx}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{L(1-s,\bar{\chi})} ds \\ &= \frac{\sqrt{2\pi}}{\tau(\chi)} \sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2 L(1+k,\bar{\chi})} \left(\frac{qx}{2\pi}\right)^{-k} \\ &+ \frac{\sqrt{2\pi}}{\tau(\chi)} \left(\int_{-\infty-iT}^{c-iT} + \int_{-\infty+iT}^{c+iT} \right) \left(\frac{qx}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{L(1-s,\bar{\chi})} ds.\end{aligned} \tag{1.60}$$

Using Lemma 1.9 and equation (A.7) we obtain the bounds for $\int_{-\infty-iT}^{c-iT}$ and $\int_{-\infty+iT}^{c+iT}$ of the form (1.55). Using Lemmas 1.9 and 1.15 we obtain the bound for the horizontal integral of (1.58) which is of the form (1.56). Combining (1.57), (1.58), (1.59) and (1.60) we conclude the proof.

1.4 Proof of Theorem 1.3

i) By repeating a similar argument as in the previous proof we deduce that if $d_F = q_F = 1$ then $F(s) = \zeta(s)$. This case is already sketched in [HL18] and the missing ingredient comes from the definition of the K class which allows us to get rid of the far left and horizontal integrals in the path of integration.

ii) In this case we consider F to be a Selberg L-function of degree $d_F = 1$ and conductor $q_F \geq 2$. Using Lemma 1.12 we find $F(s) = L(s,\chi)$ for some Dirichlet primitive character mod q_F. Therefore the completed L-function of F contains only one gamma factor and hence $r_j = 0$ or $1/2$. Since ν is real we have $\mathrm{Im}(H_F(1)) = 0$ and hence $H_F(1) = -1$ or $H_F(1) = 0$.
Suppose $H_F = -1$, then $\nu = -1/2$ and χ is an even primitive Dirichlet character mod q_F. Thus $\varphi, \psi \in K(\omega, \alpha)$ is a pair of cosine reciprocal functions. For $1 < \lambda < 1 + \delta$ and $-1 < c < 0$ we consider the positively oriented closed contour $\Omega = [\lambda - iT, \lambda + iT, c + iT, c - iT]$ where $T > 0$. Recall that the functions Z_1 and Z_2 both have a simple pole at $s = 0$. Hence from (1.42) and (1.43) we find that Φ and Ψ are analytic at $s = 0$. Furthermore, by the residue theorem

$$\frac{1}{2\pi i} \oint_{\Omega} x^{-s} Z_1(s) ds = \operatorname*{res}_{s=0} x^{-s} Z_k(s) = 2^{3/4} \Phi(0),$$

and

$$\frac{1}{2\pi i} \oint_{\Omega} x^{-s} Z_2(s) ds = \operatorname*{res}_{s=0} x^{-s} Z_k(s) = 2^{3/4} \Psi(0).$$

By the use of the bound in Lemma 1.9 and Stirling's formula for $\Gamma(s)$ the integrals along the horizontal lines of the contour Ω tend to zero as $T \to \infty$. Since (1.40) and (1.41) hold for $\lambda > 1$ we have the following cases

$$\frac{1}{2\pi i}\int_{(c)} x^{-s} Z_k(s)ds = \begin{cases} \varphi(x) - 2^{3/4}\Phi(0), & \text{if } k=1, \\ \psi(x) - 2^{3/4}\Psi(0), & \text{if } k=2. \end{cases} \tag{1.61}$$

Let $q_F := q$. If χ is an even primitive character of modulus q then $L(s,\chi)$ satisfies the functional equation

$$\frac{1}{L(1-s,\chi)} = \frac{\tau(\bar{\chi})}{q^{1/2}}\left(\frac{q}{\pi}\right)^{1/2-s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\frac{1}{L(s,\bar{\chi})}$$

for all complex values s. If we use the fact that $ab = 2\pi$ and couple this equation with (1.42), (1.43) and the functional equation of Φ and Ψ in Lemma 1.9, then we obtain

$$\frac{1}{2\pi i}\oint_{\Omega}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = \frac{1}{2\pi i}\oint_{\Omega}\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds. \tag{1.62}$$

By absolute convergence, with $c = \mathrm{Re}(s) < 0$, we may write

$$\begin{aligned} \frac{1}{2\pi i}\int_{(c)}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds &= \frac{1}{2\pi i}\int_{(c)}\left(\frac{a}{q^{1/2}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{(c)}\left(\frac{a}{q^{1/2}n}\right)^{-s} Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \frac{2^{3/4}\Phi(0)}{L(1,\chi)}, \end{aligned}$$

where we have used the case $k=1$ of (1.61). Similarly, with $\lambda = \mathrm{Re}(s) > 1$, we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{(\lambda)}\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds \\ &= \frac{1}{2\pi i}\int_{(\lambda)}\frac{\tau(\bar{\chi})b^s}{(2\pi)^{1/2}q^{s/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^s}Z_2(1-s)ds \\ &= \frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{(1-\lambda)}\left(\frac{b}{q^{1/2}n}\right)^{-w} Z_2(w)dw \\ &= \frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right) - \frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}}\frac{2^{3/4}\Psi(0)}{L(1,\bar{\chi})}, \end{aligned}$$

by making the change $w = 1-s$ and using the case $k=2$ of (1.61). Now, we may use either side of (1.62) to evaluate the residues:

- for the non-trivial zeros ρ of $L(s,\chi)$, which we assume are all simple, we have

$$\sum_{\rho}\operatorname*{res}_{s=\rho}\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})} = \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})};$$

- at $s = 1$ we have a simple pole coming from the $Z_2(1-s)$ function

$$\operatorname*{res}_{s=1} \frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^s \frac{Z_2(1-s)}{L(s,\bar\chi)} = -\frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\frac{b}{q^{1/2}}\frac{2^{3/4}\Psi(0)}{L(1,\bar\chi)};$$

- at $s = 0$ we have a trivial and simple zero of $L(s,\bar\chi)$ and we know that $Z_2(1-s)$ is analytic and non zero, so

$$\operatorname*{res}_{s=0} \frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^s \frac{Z_2(1-s)}{L(s,\bar\chi)} = \frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\frac{Z_2(1)}{L'(0,\bar\chi)} = \frac{2^{3/4}\Phi(0)}{L(1,\chi)},$$

where we have used Lemma 1.16 with $N = q$ and $k = 2$ in the last equality. Consequently, by the residue theorem we have

$$\begin{aligned}\frac{\tau(\bar\chi)b}{(2\pi)^{1/2}q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right) - \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right)\\ = \frac{\tau(\bar\chi)}{(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar\chi)}.\end{aligned}$$

Multiplying both sides by $-\sqrt{a}\sqrt{\tau(\chi)}$ and using the fact that $q^{1/2} = \sqrt{\tau(\chi)\tau(\bar\chi)}$ we have the desired result for even characters

$$\begin{aligned}&\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar\chi)}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)\\ &= -q^{1/2}\frac{\sqrt{\tau(\bar\chi)}}{b^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar\chi)}.\end{aligned} \tag{1.63}$$

We note that if we had used the other side of (1.62) instead, then the result would have been

$$\begin{aligned}&\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar\chi)}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right)\\ &= q^{1/2}\frac{\sqrt{\tau(\chi)}}{a^{1/2}}\sum_{\rho}\left(\frac{a}{q^{1/2}}\right)^{\rho}\frac{Z_1(1-\rho)}{L'(\rho,\chi)}.\end{aligned} \tag{1.64}$$

We denote by $\rho = \beta + i\gamma$ a non-trivial zero of $L(s,\bar\chi)$ and we choose $T > 0$ to tend to infinity through values such that $|T-\gamma| > \exp(-A_1|\gamma|/\log|\gamma| + 3)$ for every ordinate γ of a zero of $L(s,\chi)$. Using

$$\log|L(s,\chi)| \geqslant \sum_{|t-\gamma|\leqslant 1}\log|t-\gamma| + O(\log(qt))$$

yields

$$\log|L(\sigma+iT,\chi)| \geqslant -\sum_{|T-\gamma|\leqslant 1}A_1\gamma/\log\gamma + O(\log qT) > -A_\chi T, \tag{1.65}$$

where $A_\chi < \omega$ if A_1 is small enough and $T > T_0$. Since the main technique behind the proofs of explicit formulae is contour integration, this will enable us to make unwanted horizontal integrals tend to zero as $T \to \infty$ through the above values. To prove that indeed these horizontal integrals tend to zero as $T \to \infty$ for the chosen values we note that from (1.65) we obtain

$$\frac{1}{|L(1-s,\chi)|} \ll \exp(A_\chi T)$$

where $A_\chi < \omega$. Then by Lemma 1.9 and Stirling's formula for $\Gamma(s)$ one gets

$$\frac{1}{2\pi i}\int_{\lambda-iT}^{c-iT}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds \ll \exp\left((A_\chi-\omega+\epsilon)|t|\right) \to 0$$

for each $\epsilon > 0$. This could alternatively be proved by using Remark 1.10. The other horizontal integral is dealt with similarly.
Let us now consider $H_F = 0$, then $\nu = 1/2$ and χ is an odd primitive Dirichlet character mod q_F. Therefore $\varphi, \psi \in K(\omega, 0, -\delta)$ is a pair of sine reciprocal functions. Note Z_1 and Z_2 are both analytic at $s = 1$ hence Φ and Ψ both analytic at $s = 1$. Then by the functional equation in (1.9) we see Φ and Ψ are both analytic at $s = 0$. Thus both Z_1 and Z_2 are analytic at $s = 0$. Similarly as (1.61) we find

$$\frac{1}{2\pi i}\int_{(c)} x^{-s}Z_k(s)ds = \begin{cases}\varphi(x), & \text{if } k=1,\\ \psi(x), & \text{if } k=2.\end{cases} \tag{1.66}$$

Let $q := q_F$. If χ is an odd, primitive and non-principal character of mod q then $L(s,\chi)$ satisfies the functional equation

$$\frac{1}{L(1-s,\chi)} = \frac{\tau(\bar{\chi})}{iq^{1/2}}\left(\frac{q}{\pi}\right)^{1/2-s}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\frac{1}{L(s,\bar{\chi})},$$

for all complex values s. If we use the fact that $ab = 2\pi$ and couple this equation with (1.33), (1.34) and the functional equation of Φ and Ψ in Lemma 1.9, then we obtain

$$\frac{1}{2\pi i}\oint_{\Omega}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = \frac{1}{2\pi i}\oint_{\Omega}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds.$$

By absolute convergence with $\mathrm{Re}(s) = c$ we can change summation and integration to obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{(c)}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds &= \frac{1}{2\pi i}\int_{(c)}\left(\frac{a}{q^{1/2}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}Z_1(s)ds\\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{(c)}\left(\frac{a}{q^{1/2}n}\right)^{-s}Z_1(s)ds\\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right),\end{aligned} \tag{1.67}$$

where in ultimate step we have used (1.66) with $k = 1$. Moreover, also by absolute convergence with $\mathrm{Re}(s) = \lambda$, we have

$$\frac{1}{2\pi i}\int_{(\lambda)}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds$$

$$
\begin{aligned}
&= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \frac{1}{2\pi i} \int_{(\lambda)} \left(\frac{b}{q^{1/2}}\right)^s \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^s} Z_2(1-s)ds \\
&= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \frac{b}{q^{1/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \frac{1}{2\pi i} \int_{(1-\lambda)} \left(\frac{b}{q^{1/2}n}\right)^{-w} Z_2(w)dw \\
&= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \frac{b}{q^{1/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right),
\end{aligned}
$$

where we have made the change $w = 1 - s$. A similar reasoning as the one we used for even primitive characters shows that the contribution from the horizontal integrals of this contour will tend to zero as well. Next, we compute the residues:

- for the non-trivial zeros ρ one has

$$
\sum_{\rho} \operatorname*{res}_{s=\rho} \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \left(\frac{b}{q^{1/2}}\right)^s \frac{Z_2(1-s)}{L(s,\bar{\chi})} = \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.
$$

By the residue theorem one has

$$
\begin{aligned}
&\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \frac{b}{q^{1/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right) - \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \varphi\left(\frac{a}{q^{1/2}n}\right) \\
&= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.
\end{aligned}
$$

Multiplying by $-\sqrt{a}\sqrt{\tau(\chi)}$ and using the fact that $\sqrt{\tau(\chi)\tau(\bar{\chi})} = iq^{1/2}$ one has

$$
\begin{aligned}
&\sqrt{a}\sqrt{\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right) \\
&= -\frac{q^{1/2}}{b^{1/2}} \sqrt{\tau(\bar{\chi})} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\rho)}{L'(\rho,\bar{\chi})} Z_2(1-\rho), \qquad (1.68)
\end{aligned}
$$

and this proves the theorem.

1.5 Proof of Theorem 1.6

A similar argument as in the beginning of the proof of Theorem 1.1 yields $F(s) = \zeta(s)$ and $\nu = -1/2$. Therefore $Z_1(s)$ is meromorphic with simple poles at $s = 0, -2, -4, \cdots$. Thus for $0 < c < 1$ we define

$$
W(x) := \frac{1}{2\pi i} \int_{(c)} \frac{Z_1(-s)}{\zeta(1+s)} x^s ds.
$$

By using the fact that $c > 0$ we can write

$$
W(x) = \frac{1}{2\pi i} \int_{(c)} Z_1(-s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+s}} x^s ds = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{2\pi i} \int_{(c)} Z_1(-s) \left(\frac{n}{x}\right)^{-s} ds.
$$

The change of variable $w = -s$ yields

$$\begin{aligned} W(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{2\pi i} \int_{(-c)} Z_1(w) \Big(\frac{x}{n}\Big)^{-w} dw = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \varphi\left(\frac{x}{n}\right) - \operatorname*{res}_{w=0} Z_1(w) \Big(\frac{x}{n}\Big)^{-w} \right\} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left\{ \varphi\left(\frac{x}{n}\right) - 2^{3/4}\Phi(0) \right\} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \varphi\left(\frac{x}{n}\right) = P_\varphi(x), \end{aligned}$$

where in the second line we have used the fact that $-1 < -c < 0$ and the prime number theorem on the fourth line. By the theory of Mellin transforms we obtain

$$\Upsilon(s) := \int_0^\infty P_\varphi(x) x^{-s-1} dx = \frac{Z_1(-s)}{\zeta(1+s)}. \tag{1.69}$$

Therefore, multiplying both sides by s we have that

$$s\zeta(1+s)\Upsilon(s) = sZ_1(-s), \tag{1.70}$$

for $0 < \mathrm{Re}(s) < 1$. Now we will study (1.69) for $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. To do this, we split the integral representation of $\Upsilon(s)$ at x =1 and apply the bound $P_\varphi(x) \ll x^{-\frac{1}{2}+\delta}$ for any $\delta > 0$ as $x \to \infty$ so that

$$\begin{aligned} \Upsilon(s) &= \int_0^1 P_\varphi(x) x^{-s-1} dx + \int_1^\infty P_\varphi(x) x^{-s-1} dx = O(1) + O\bigg(\int_1^\infty x^{-\frac{1}{2}+\delta} x^{-\sigma-1} dx \bigg) \\ &= O(1). \end{aligned}$$

Thus one can now see that the application of the bound $P_\varphi(x) \ll x^{-\frac{1}{2}+\delta}$ makes the integral analytic on the interval $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. We reason as follows. Since the simple pole of $\zeta(1+s)$ and $Z_1(-s)$ is annihilated by the zero of s at $s = 0$ we see that the left-hand side of (1.70) is analytic. Since (1.70) holds for $0 < \mathrm{Re}(s) < 1$, by the theory of analytic continuation, it also holds on $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. If $Z_1(-s)$ does not have any zeros in the interval $-\frac{1}{2} < \mathrm{Re}(s) \le 0$, then the left-hand side of (1.70) is non-zero in $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. However, since $\Upsilon(s)$ has been shown to be analytic in this interval when the bound on $P_\varphi(x)$ is applied, this implies that $\zeta(1+s)$ does not have zeros in $-\frac{1}{2} < \mathrm{Re}(s) \le 0$. This implies the Riemann hypothesis.
If $Z_1(-s)$ actually had zeros then all the zeros of the Riemann zeta-function would still lie on the critical line except for the zeros that coincide with the zeros of $Z_1(-s)$.

Let us now prove that the Riemann hypothesis implies the bound $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$. We recall a formulation of the Riemann hypothesis involving Mertens's function due to Littlewood [Lit12] which says that

$$M(x) \ll x^{\frac{1}{2}+\varepsilon}.$$

An application of partial summation allows us to transform this into

$$M(\nu, n) := \sum_{m=\nu}^{n} \frac{\mu(m)}{m} \ll_\varepsilon \nu^{-\frac{1}{2}+\varepsilon} \tag{1.71}$$

uniformly in n. Recalling the definition of P_φ we have

$$P_\varphi(y) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \varphi\left(\frac{y}{n}\right) = \left(\sum_{n=1}^{\nu-1} + \sum_{n=\nu}^{\infty}\right) \frac{\mu(n)}{n} \varphi\left(\frac{y}{n}\right) =: P_{\varphi,1}(y) + P_{\varphi,2}(y),$$

where $\nu = \lfloor y^{1-\varepsilon} \rfloor$. We handle each sum separately. For the first sum

$$P_{\varphi,1}(y) = \sum_{n=1}^{\nu-1} \frac{\mu(n)}{n} \varphi\left(\frac{y}{n}\right) \ll \sum_{n=1}^{\nu-1} \frac{e^{-y/n}}{n},$$

since $\varphi \in K(\omega, 0)$ and where we have used the asymptotic of φ for large y. Therefore,

$$P_{\varphi,1}(y) \ll ye^{-y}. \tag{1.72}$$

For the second sum, we have

$$\begin{aligned} P_{\varphi,2}(y) &= \sum_{n=\nu}^{\infty} \frac{\mu(n)}{n} \varphi\left(\frac{y}{n}\right) = \sum_{n=\nu}^{\infty} M(\nu, n) \left\{\varphi\left(\frac{y}{n}\right) - \varphi\left(\frac{y}{n+1}\right)\right\} \\ &= \sum_{n=\nu}^{\infty} M(\nu, n) \left\{-\frac{y}{\lambda_n^2} \varphi'\left(\frac{y}{\lambda_n}\right)\right\}, \\ &\ll \nu^{-\frac{1}{2}+\varepsilon} \left(\sum_{n=\nu}^{\beta-1} + \sum_{n=\beta}^{\infty}\right) \left|\frac{y}{\lambda_n^2} \varphi'\left(\frac{y}{\lambda_n}\right)\right| \\ &=: P_{\varphi,3}(y) + P_{\varphi,4}(y), \end{aligned} \tag{1.73}$$

where in the last line we have used the mean value theorem with $a = n < \lambda_n (= c) < n + 1 = b$ and where

$$P_{\varphi,3}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \sum_{n=\nu}^{\beta-1} \left|\frac{y}{\lambda_n^2} \varphi'\left(\frac{y}{\lambda_n}\right)\right|, \quad P_{\varphi,4}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \sum_{n=\beta}^{\infty} \left|\frac{y}{\lambda_n^2} \varphi'\left(\frac{y}{\lambda_n}\right)\right|$$

with $\beta = \lfloor y^{1+\varepsilon} \rfloor$. We start with $P_{\varphi,4}(y)$ first. By the definition of the class K and by Cauchy's integral formula we see that

$$\varphi'\left(\frac{y}{\lambda_n}\right) \ll e^{-y/\lambda_n}$$

for $\lambda_n \geq \beta$. Thus

$$P_{\varphi,4}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \sum_{n=\beta}^{\infty} \left|\frac{y}{\lambda_n^2} e^{-y/\lambda_n}\right| \ll \nu^{-\frac{1}{2}+\varepsilon} e^{-y/\beta} \beta^{-1+(\delta+\varepsilon)} y \sum_{n=\beta}^{\infty} \frac{1}{\lambda_n^{1+\varepsilon}} \ll y^{-\frac{1}{2}+\varepsilon'}. \tag{1.74}$$

For the sum $P_{\varphi,3}$ we reason as follows. First, φ is analytic, thus φ' is continuous in a compact interval containing $I(\varepsilon, y) = (y^{-\varepsilon}, y^{\varepsilon}) \subset [0, y]$. Therefore, there exists a point $c \in I(\varepsilon, y)$ such that

$$\varphi'(c) = \max_{[0,y]} \varphi'(x).$$

The value of c is independent of y. To see this, note that $\varphi'(x) \ll e^{-x}$ when $x \to \infty$. Then we can find a positive real number C independent of y such that $\varphi'(c) > \varphi'(y)$ for all $y > C$. Therefore,

$$P_{\varphi,3}(y) \ll \nu^{-\frac{1}{2}+\varepsilon} \frac{y}{\nu^2} \varphi'(c) \sum_{n=\nu}^{\beta-1} 1 \ll y^{-\frac{1}{2}+\varepsilon''}. \tag{1.75}$$

Putting together (1.72), (1.73), (1.74) and (1.75) we see that the Riemann hypothesis implies the bound $P_\varphi(y) \ll y^{-\frac{1}{2}+\delta}$ as $y \to \infty$ for all $\delta > 0$.

Chapter 2

Moments of the average of a generalized Ramanujan sum

2.1 Introduction

In [Ram18], Ramanujan introduced the following trigonometrical sum.

Definition 2.1. The Ramanujan sum $c_q(n)$ is defined by

$$c_q(n) := \sum_{(h,q)=1} \cos\left(\frac{2\pi nh}{q}\right) = \sum_{(h,q)=1} e^{2\pi inh/q}, \tag{2.1}$$

where q and n are positive integers and the summation is taken over a reduced residue system mod q.

Ramanujan sums fit naturally with other arithmetical functions. For instance, one has

$$c_q(1) = \mu(q) \quad \text{and} \quad c_q(q) = \phi(q),$$

where $\mu(n)$ and $\phi(n)$ are the Möbius and Euler totient functions, respectively. Moreover, if $(q,r) = 1$, then $c_q(n)c_r(n) = c_{qr}(n)$, i.e. it is a multiplicative function. In the same article, Ramanujan obtained expressions of the form

$$f(n) = \sum_{q=1}^{\infty} a_q c_q(n) \tag{2.2}$$

for some coefficients a_q. In particular,

$$\sum_{q=1}^{\infty} \frac{c_q(n)}{q} = 0, \quad \sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n) = -d(n), \quad \sum_{q=1}^{\infty} \frac{(-1)^{q-1} c_{2q-1}(n)}{2q-1} = \frac{1}{\pi} r(n) \tag{2.3}$$

as well as

$$\sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}} = \frac{\sigma_{-s}(n)}{\zeta(s+1)} \quad \text{and} \quad \sum_{q=1}^{\infty} \frac{\mu(q)c_q(n)}{\phi_{s+1}(q)} = \zeta(s+1)\frac{\phi_s(n)}{n^s} \tag{2.4}$$

for $\mathrm{Re}(s) > 0$. Here $d(n)$ is the number of divisor of n, $\sigma_s(n)$ the sum of their s-th powers,

$$\phi_s(n) = n^s \left(1 - \frac{1}{p_1^s}\right)\left(1 - \frac{1}{p_2^s}\right)\cdots\left(1 - \frac{1}{p_k^s}\right)$$

when $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, and $r(n)$ is the number of representations of n as the sum of two squares. Also he showed that

$$\sum_{n=1}^{\infty} \frac{c_q(n)}{n} = -\Lambda(q) \quad \text{and} \quad \sum_{\substack{d|n \\ d|q}} d\mu(q/d) = c_q(n), \tag{2.5}$$

where $\Lambda(n)$ is the von Mangoldt function.

The second equation of (2.3) is of the same depth as the prime number theorem. As discussed by Hardy and Wright in [HW54], these series have a particular interest because they show explicitly the source of the irregularities in the behavior of their sums. Note that the Ramanujan expansion (2.2) mimics the notion of a Fourier expansion of an L^1-function. In [Car], Carmichael noticed an orthogonality principle of Ramanuajn sums. This allows one to predict the Ramanujan coefficients a_q in (2.2) of an arithmetical function $f(n)$ if such expansion exists. The work of Wintner [Win13] and Delange [Del76] allows us to determine a large number of Ramanujan expansions. Later on more work was done in this direction by Delange [Del63], Wirsing [Wir61], Hildebrand [Hil84], Schwarz [Sch88], Lucht and Reifenrath [LR01].

Ramanujan sums and their variations make surprising appearances in singular series of the Hardy-Littlewood asymptotic formula for Waring problems and in the asymptotic formula of Vinogradov on sums of three primes, for details the reader is referred to [Dav00].

Recently, Alkan [Alk12] studied the weighted averages of Ramanujan sums. He showed that for integer $r \geq 1$ and $x \geq 1$ one has

$$\begin{aligned} \sum_{k \leq x} \left(\frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j) \right) &= 1 + \frac{1}{2} \sum_{2 \leq k \leq x} \frac{\phi(k)}{k} \\ &\quad + \frac{1}{1+r} \sum_{m=1}^{[\frac{r}{2}]} \binom{r+1}{2m} B_{2m} \sum_{2 \leq k \leq x} \prod_{p|k} \left(1 - \frac{1}{p^{2m}}\right), \end{aligned} \tag{2.6}$$

where $B_{2m} \neq 0$ are the Bernoulli numbers together with the convention that the sum over m is taken to be zero when $r = 1$ and the sums over k are taken to be zero when $1 \leq x < 2$.

In [CK12], Chan and Kumchev studied moments of averages of Ramanujan sums. They showed that for $y \geq x$ one has

$$\sum_{n \leq y} \sum_{q \leq x} c_q(n) = y - \frac{x^2}{4\zeta(2)} + O(xy^{1/3} \log x + x^3 y^{-1}), \tag{2.7}$$

as well as

$$\sum_{n\le y}\left(\sum_{q\le x} c_q(n)\right)^2 = \frac{yx^2}{2\zeta(2)} + O(x^4 + xy\log x) \tag{2.8}$$

for $y \ge x^2(\log x)^B$ for $B > 0$, and lastly for $x \le y \le x^2(\log x)^B$

$$\sum_{n\le y}\left(\sum_{q\le x} c_q(n)\right)^2 = \frac{yx^2}{2\zeta(2)}(1+2\kappa(u)) + O(yx^2(\log x)^{10}(x^{-1/2} + (y/x)^{-1/2})), \tag{2.9}$$

where $u = \log(yx^{-2})$ and $\kappa(u)$ is a certain Fourier integral given by

$$\kappa(u) := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(it)e^{-itu}dt, \quad \text{where} \quad f(s) := \frac{\zeta(1-s)}{\zeta(1+s)}\frac{1}{(1+s)^2(1-s)}. \tag{2.10}$$

It satisfies some numerical inequalities given in [CK12] and in particular $\kappa(u) = o(1)$.

Let β be a positive integer. A generalization of the Ramanujan sum due to Cohen [Coh49] is written as $c_{q,\beta}(n)$ and it is defined by

$$c_{q,\beta}(n) := \sum_{(h,q^\beta)_\beta=1} e^{2\pi i n h/q^\beta}, \tag{2.11}$$

where h ranges over the the non-negative integers less than q^β such that h and q^β have no common β-th power divisors other than 1. It follows immediately that when $\beta = 1$, (2.11) becomes the Ramanujan sum (2.1). Clearly this generalization of Ramanujan sum is as important as the Ramanujan sum by its arithmetic nature. For more arithmetic properties of the generalized Ramanuajan sum (2.11), the reader is referred to [Coh49]. For a discussion of the connections between the generalized Ramanujan sums due to Cohen and the non-trivial zeros of the Riemann zeta-function the reader is referred to [KR15].

Let us now introduce the main object of study of this chapter. The k^{th} moment of the average of the generalized Ramanujan sum (2.11) is defined by

$$C_{k,\beta}(x,y) := \sum_{n\leqslant y}\left(\sum_{q\leqslant x} c_{q,\beta}(n)\right)^k, \tag{2.12}$$

where k is a positive integer and x and y are reals. It is not to difficult to obtain an asymptotic result for the k^{th} moment of (2.12). In particular we have

Proposition 2.2. *Let k and β be two positive integers. Let $y > x^{k(1+\beta)}\log^{k+1} x$, then*

$$C_{k,\beta}(x,y) = A_{k,\beta}(x,y) + O(x^{k(1+\beta)}\log^k x), \tag{2.13}$$

where

$$A_{k,\beta}(x,y) = \begin{cases} y, & \text{if } k=1, \\ \frac{yx^{1+\beta}}{(1+\beta)\zeta(1+\beta)} + O(yx^\beta \log^{\lfloor 1/\beta\rfloor} x), & \text{if } k>1. \end{cases} \tag{2.14}$$

For the first and second moments one can improve the error terms as well as clarify the dependence between the parameters y and x. Our main results are following.

Theorem 2.3. *Let $y \geq x^{3\beta/2} \log^5 x$. Then for $\beta = 1, 2$ one has*

$$C_{1,\beta}(x,y) = y - \frac{x^{1+\beta}}{2(1+\beta)\zeta(1+\beta)} + O(x^\beta y^{1/3} \log^4 y + x^{2\beta+1} y^{-2/3} + x^{\beta+1} y^{-1/3}). \tag{2.15}$$

and for $\beta \geq 3$ one has

$$C_{1,\beta}(x,y) = y + O(x^\beta y^{1/3} \log^4 y). \tag{2.16}$$

Theorem 2.4. *For $\beta = 1, 2$ and $x^{2\beta} < y < x^{2\beta+\beta^2} \log^{\frac{5}{2}(\beta+1)} x$ one has*

$$\begin{aligned} C_{2,\beta}(x,y) = {} & \frac{yx^{1+\beta}}{(1+\beta)\zeta(1+\beta)} - \frac{1}{2} \frac{x^{2+2\beta}}{(1+\beta)^2 \zeta^2(1+\beta)} \\ & + O(y^{-1} x^{2+4\beta} + x^{2\beta+1}(\log x + \log\log x)) \\ & + O(x^{2\beta} y^{\frac{1}{3}+\frac{1}{6\beta}} (\log^5 y) \log\log y (\log^4 x + \log^4 \log x) \\ & \qquad + yx^{\frac{1}{2}+\beta}(\log^3 x + \log^3 \log x)). \end{aligned} \tag{2.17}$$

For $\beta \geq 3$ and $y > x^{3\beta/2}$ one has

$$\begin{aligned} C_{2,\beta}(x,y) = {} & \frac{yx^{1+\beta}}{(1+\beta)\zeta(1+\beta)} + O(x^{2\beta} y^{\frac{1}{3}+\frac{1}{6\beta}} (\log^5 y) \log\log y (\log^4 x + \log^4 \log x)) \\ & + O(yx^{\frac{1}{2}+\beta}(\log^3 x + \log^3 \log x)). \end{aligned} \tag{2.18}$$

Let us note the following remarks.

(i) Theorem 2.4 is not only more general but also improves the error term in (2.9) when we choose $\beta = 1$.

(ii) In order to improve Proposition 2.2 for $k \geq 3$ one may need to assume some strong results such as the moment hypothesis of the Riemann zeta-function. For $k \geq 3$ improving Proposition 2.2 unconditionally is still an open question.

It is worth remarking (see [CK12]) that the introduction of van der Corput's method of exponential sums leads to the following result concerning the first moment.

Theorem 2.5. *Let $\beta \in \mathbb{N}$ be fixed. Let x be a large real and $y \geq x^\beta$. One has*

$$C_{1,\beta}(x,y) = y - \frac{x^{1+\beta}}{2(1+\beta)\zeta(1+\beta)} + R_{1,\beta}(x,y), \tag{2.19}$$

where

$$R_{1,\beta}(x,y) \ll_\beta \begin{cases} xy^{\frac{1}{3}} \log x + x^3 y^{-1}, & \text{if } \beta = 1, \\ x^{\frac{1+2\beta}{3}} y^{\frac{1}{3}} + x^{1+2\beta} y^{-1}, & \text{if } \beta > 1. \end{cases} \tag{2.20}$$

We remark that the range of y is different than the one in Theorem 2.3 and that when $\beta = 1$, (2.7) follows as a special case.

Next we consider a generalization of the divisor function, defined by

$$\sigma_{z,\beta}(n) := \sum_{d^\beta | n} d^{\beta z}. \tag{2.21}$$

Crum [Cru40] seems to be first author who coined the notation (2.21). Understanding the asymptotic behavior of sums like

$$\sum_{n \le x} \sigma_{z_1,\beta}(n) \quad \text{and} \quad \sum_{n \le x} \sigma_{z_1,\beta}(n)\sigma_{z_2,\beta}(n)$$

for $\mathrm{Re}(z_i) \le 0$, $i = 1, 2$ is naturally needed in the proofs of Theorems 2.3 and 2.4. However, these sums are important objects in their own right. Clearly these sums are generalizations of

$$\sum_{n \le x} d(n) \quad \text{and} \quad \sum_{n \le x} d^2(n),$$

respectively.

The evaluation of the summation of the divisor function

$$D(x) := \sum_{n \le x} d(n),$$

has been studied extensively in the literature. In particular, it can be shown that

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x),$$

and the specific determination of the error term $\Delta(x)$ is called the Dirichlet divisor problem (see [MV07, p. 68]). In 1849, Dirichlet [Dir49] proved that $\Delta(x)$ could be taken to be $O(x^{1/2})$. Further progress came in 1903 by Voronoï [Vor03], who showed that $\Delta(x) \ll x^{1/3} \log x$, and then by van der Corput who proved in 1922 that $\Delta(x) \ll x^{33/100+\varepsilon}$, [vdC22]. The exponent has been reduced over the years (see [MV07, p. 69] for further details). The current record stands at $\Delta(x) \ll x^{131/416+\varepsilon}$ and it is due to Huxley [Hux03].

On the other hand, in [Ram16] Ramanujan states without proof that

$$d^2(1) + d^2(2) + \cdots + d^2(n) = An(\log n)^3 + Bn(\log n)^2 + Cn \log n + Dn + O(n^{3/5+\varepsilon}). \tag{2.22}$$

Moreover, and also without proof, Ramanujan claims that on the Riemann hypothesis, the error term in (2.22) can be strengthened to $O(n^{1/2+\varepsilon})$. In 1922, Wilson [Wil] proved that indeed one can take the error term to be $O(n^{1/2+\varepsilon})$ unconditionally. As can be seen from [MV81, Ram00], it is highly probable that that the error term is $O(n^{1/2})$. Suppose that $P_3(t)$ denotes a polynomial in t of degree 3. Let us set the notation

$$E(x) := \sum_{n \le x} d^2(n) - xP_3(\log x).$$

Ramachandra and Sankaranarayanan [RS03] proved that

$$E(x) = O(x^{1/2}(\log x)^5(\log\log x))$$

unconditionally. In 1962, Chandrasekharan and Narasimhan [CN61] showed that

$$E(x) = \Omega_{\pm}(x^{1/4}).$$

For a given arithmetic function $f(n)$ we define

$$\sum_{n\le x}{}' f(n) = \sum_{n\leqslant x} f(n) - \frac{1}{2} f(x),$$

when x is a positive integer. We have following asymptotic results.

Theorem 2.6. *Let* $\mathrm{Re}(z) \le 0$. *Then*

$$\sum_{n\leqslant x}{}' \sigma_{z,\beta}(n) = D_{z,\beta}(x) + \Delta_{z,\beta}(x),$$

where

$$\Delta_{z,\beta}(x) \ll x^{\frac{1}{3}} \log^2 x$$

uniformly for $\beta \ge 1$ *and* $D_{z,\beta}(x)$ *is given by following.*

(i) *If* $\beta = 1, 2$ *and* $-\frac{2}{3\beta^2} < \mathrm{Re}(z) \le 0$, *then*

$$D_{z,\beta}(x) = \zeta(\beta(1-z))x + \frac{1}{1+\beta z}\zeta\left(z + \frac{1}{\beta}\right) x^{z+\frac{1}{\beta}}. \tag{2.23}$$

(ii) *If* $\beta \ge 3$ *and* $-1 < \mathrm{Re}(z) \le 0$, *then*

$$D_{z,\beta}(x) = \zeta(\beta(1-z))x. \tag{2.24}$$

Theorem 2.7. *Let* $\mathrm{Re}(z_1), \mathrm{Re}(z_2) \le 0$, $\mathrm{Re}(z_1+z_2) > -1$ *and* $|\,\mathrm{Re}(z_1 - z_2)| < 1/b$. *Then for* $\beta \ge 1$ *one has*

$$\sum_{n\leqslant x}{}' \sigma_{z_1,\beta}(n)\sigma_{z_2,\beta}(n) = D_{z_1,z_2,\beta}(x) + \Delta_{z_1,z_2,\beta}(x), \tag{2.25}$$

where the values $D_{z_1,z_2,\beta}(x)$ *and* $\Delta_{z_1,z_2,\beta}(x)$ *are given below.*

(i) *If* $\beta = 1, 2$ *and* $-\frac{1}{2(2\beta+1)} < \mathrm{Re}(z_1), \mathrm{Re}(z_2), \mathrm{Re}(z_1+z_2) \le 0$ *then for* $z_1 \ne 0, z_2 \ne 0$, *and* $z_1 \ne z_2$ *one has*

$$\begin{aligned} D_{z_1,z_2,\beta}(x) =& \frac{\zeta(\beta(1-z_1))\zeta(\beta(1-z_2)\zeta(\beta(1-z_1-z_2))}{\zeta(\beta(2-z_1-z_2))} x \\ &+ \frac{\zeta(z_1+1/\beta)\zeta(1+\beta z_1 - \beta z_2)\zeta(1-\beta z_2)}{(\beta z_1+1)\zeta(2+\beta z_1-\beta z_2)} x^{z_1+\frac{1}{\beta}} \\ &+ \frac{\zeta(z_2+1/\beta)\zeta(1+\beta z_2 - \beta z_1)\zeta(1-\beta z_1)}{(\beta z_2+1)\zeta(2+\beta z_2-\beta z_1)} x^{z_2+\frac{1}{\beta}} \\ &+ \frac{\zeta(z_1+z_2+1/\beta)\zeta(\beta z_2+1)\zeta(\beta z_1+1)}{(\beta z_1+\beta z_2+1)\zeta(2+\beta z_1+\beta z_2)} x^{z_1+z_2+\frac{1}{\beta}}, \end{aligned} \tag{2.26}$$

and

$$\Delta_{z_1,z_2,\beta}(x) \ll x^{\frac{1}{3}+\frac{1}{6\beta}+\frac{\mathrm{Re}(z_1)+\mathrm{Re}(z_2)}{6}}(\log^5 x)\log\log x. \tag{2.27}$$

(ii) *If* $\beta \geq 3$ *then for* $z_1 \neq 0, z_2 \neq 0$, *and* $z_1 \neq z_2$, *then*

$$D(z_1,z_2,\beta)(x) = \frac{\zeta(\beta(1-z_1))\zeta(\beta(1-z_2))\zeta(\beta(1-z_1-z_2))}{\zeta(\beta(2-z_1-z_2))}x, \tag{2.28}$$

and

$$\Delta_{z_1,z_2,\beta}(x) \ll \max\left(x^{\frac{1}{2\beta}+\frac{\mathrm{Re}(z_1)+\mathrm{Re}(z_2)}{2}+\frac{\beta|\mathrm{Re}(z_1)-\mathrm{Re}(z_2)|}{3}}, x^{\frac{1}{3}+\frac{1}{6\beta}+\frac{\mathrm{Re}(z_1)+\mathrm{Re}(z_2)}{6}}\right)$$
$$\times (\log^5 x)\log\log x. \tag{2.29}$$

Remark 2.8. For other values of z_1 and z_2, such as $z_i = 0$ or $z_1 = z_2$, one can compute explicitly the value of $D_{z_1,z_2,\beta}(x)$. Since the other cases are not of interest in the present chapter, only values of $D_{z_1,z_2,\beta}(x)$ for $\mathrm{Re}(z_i) \leq 0$, $i = 1, 2$ are provided. In the proof of Theorem 2.7 we will see how the other values can, in fact, be obtained. Theorems 2.6 and 2.7 can be computed for $\mathrm{Re}(z_i) > 0$, $i = 1, 2$ by similar arguments of the methods presented here. We also avoid these cases.

The chapter is organized as follows. In the next secion, we state and prove some preliminary tools. Then, in §2.3 we prove Proposition 2.2 by elementary methods. Next, in §2.4 and §2.5, we move on to the proofs of Theorems 2.6 and 2.7, respectively, by analytic methods. Finally, in §2.6 and §2.7, we deal with the proofs of Theorems 2.3 and 2.4. The proof by van der Corput's method of Theorem 2.5 is in §2.8. Lastly, some numerical evidence and plots are given at the end.

2.2 Preliminaries

We start this section by recalling two important identities due to Cohen [Coh49]. These identities generalize the first identity of (2.4) and the second identity of (2.5).

Lemma 2.9. *Suppose that* β *is a positive integer, then one has*

$$\sum_{q=1}^{\infty} \frac{c_{q,\beta}(n)}{q^{\beta s}} = \frac{\sigma_{1-s,\beta}(n)}{\zeta(\beta s)}$$

for $\mathrm{Re}(s) > 1$.

Lemma 2.10. *The generalization of the Ramanujan sum may be written as*

$$c_{q,\beta}(n) = \sum_{\substack{d|q \\ d^\beta | n}} d^\beta \mu\left(\frac{q}{d}\right), \tag{2.30}$$

where $\mu(n)$ *is the Möbius function.*

In [Cru40], Crum derived the Dirichlet series for $\sigma_{z,\beta}(n)$.

Lemma 2.11. *Suppose that β is a positive integer and that $z \in \mathbb{C}$. One has*

$$\sum_{n=1}^{\infty} \frac{\sigma_{z,\beta}(n)}{n^s} = \zeta(s)\zeta(\beta(s-z)).$$

for $\mathrm{Re}(s) > \max(\mathrm{Re}(z) + \frac{1}{\beta}, 1)$.

The Dirichlet series for $\sigma_{p,\beta}(n)\sigma_{q,\beta}(n)$ is given in [Cru40].

Lemma 2.12. *One has*

$$\sum_{n=1}^{\infty} \frac{\sigma_{z_1,\beta}(n)\sigma_{z_2,\beta}(n)}{n^s} = \frac{\zeta(s)\zeta(\beta(s-z_1))\zeta(\beta(s-z_1))\zeta(\beta(s-z_1-z_2))}{\zeta(\beta(2s-z_1-z_2))}$$

for $\mathrm{Re}(s) > \max(1, \mathrm{Re}(z_1) + 1/\beta, \mathrm{Re}(z_2) + 1/\beta, \mathrm{Re}(z_1 + z_2) + 1/\beta)$.

From [Tit86, Lemma 4.5, page 72] we have

Lemma 2.13. *Let a, b and M be real numbers and $r > 0$. Let F be a real valued function, twice differentiable, and $|F''(x)| \geq r$ in $[a, b]$. Let G be a real valued function, G/F' be monotonic and $|G(x)| \leq M$. Then*

$$\left| \int_a^b G(x)e^{iF(x)}dx \right| \leq \frac{8M}{\sqrt{r}}.$$

Let $\mathcal{M}$ be a class of non-negative arithmetic functions which are multiplicative and that satisfy:

(i) there exists a positive constant A such that if p is a prime and $l \geq 1$ then

$$f(p^l) \leq A^l;$$

(ii) for every $\epsilon > 0$, there exists a positive constant $B(\epsilon)$ such that

$$f(n) \leq B(\epsilon)n^{\epsilon}$$

for $n \geq 1$.

In [Shi80b], Shiu showed

Lemma 2.14. *Let $f \in \mathcal{M}$, $0 < \alpha, \beta < 1/2$ and let a, k be integers. If $0 < a < k$ and $(a, k) = 1$, then as $x \to \infty$*

$$\sum_{\substack{x-y<n\leq x \\ n\equiv a \bmod q}} f(n) \ll \frac{y}{\phi(q)\log x} \exp\left(\sum_{\substack{p\leq x \\ p\nmid q}} \frac{f(p)}{p}\right),$$

uniformly in a, q, and y provided that $q \leq y^{1-\alpha}$, $x^{\beta} < y \leq x$.

In [NT98], Nair and Tenenbaum observed that if $q = 1$ then one can obtain the same result when f is non-negative, sub-multiplicative and satisfying (i) and (ii). We also recall the following well-known estimate.

Lemma 2.15. *Let B_1 be the Mertens constant*[1]*. Then*

$$\sum_{p\le x}\frac{1}{p}=\log\log x+B_1+O\left(\frac{1}{\log x}\right).$$

The following lemma can be easily adopted from [MV07, Theorem 5.2]. For the sake of completeness we will give the sketch of the proof.

Lemma 2.16. *Let $0<\lambda_1<\lambda_2<\cdots<\lambda_n\to\infty$ be any sequence of real numbers and let $\{a_n\}$ be any sequence of complex numbers. Let the Dirichlet series $\alpha(s):=\sum_{n=1}^{\infty}a_n\lambda_n^{-s}$ be absolutely convergent for some $\mathrm{Re}(s)>\sigma_a$. If $\sigma_0>\max(0,\sigma_a)$ and $x>0$, then*

$$\sum_{\lambda_n\le x}{}' a_n=\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}ds+R,$$

where

$$R\ll\sum_{\substack{x/2<\lambda_n<2x\\ n\ne x}}|a_n|\min\left(1,\frac{x}{T|x-\lambda_n|}\right)+\frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{\lambda_n^{\sigma_0}}.$$

Proof. Let

$$\mathrm{si}(x):=-\int_x^{\infty}\frac{\sin u}{u}du. \tag{2.31}$$

Integrating by parts one obtains

$$\mathrm{si}(x)\ll\min(1,1/x) \tag{2.32}$$

for $x>0$. The proof of the lemma follows from the following identity [MV07, p. 139, Eq. (5.9)]

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{x^s}{s}ds\ll\begin{cases}1+O(\frac{x^{\sigma_0}}{T}), & \text{if } x\ge 2,\\ 1+\frac{1}{\pi}\,\mathrm{si}(T\log x)+O(\frac{2^{\sigma_0}}{T}), & \text{if } 1\le x\le 2,\\ -\frac{1}{\pi}\,\mathrm{si}(-T\log x)+O(\frac{2^{\sigma_0}}{T}), & \text{if } \frac{1}{2}\le x\le 1\\ O(\frac{x^{\sigma_0}}{T}), & \text{if } x\le\frac{1}{2}\end{cases} \tag{2.33}$$

for $\sigma_0>0$. Now

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}ds=\sum_{n=1}^{\infty}a_n\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\frac{(x/\lambda_n)^s}{s}ds. \tag{2.34}$$

Applying (2.32), (2.33) and the fact that

$$|\log(1+\delta)|\asymp|\delta|$$

[1] The existence of Mertens' constant is related to Mertens' second theorem which states that

$$\lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p}-\log\log x-B_1=0\right).$$

The numerical value is $B_1\approx 0.2614972\cdots$

in (2.34) we obtain the desired result. □

Note that for any fixed real number t' and $\sigma \geq 1/2$ (see [Tit86, Chap. VII])

$$\int_{T/2}^{T} |\zeta^4(\sigma + i(t+t'))| dt \ll T \log^4 T.$$

Therefore arguing in a similar fashion as in [RS03, Lemmas 3.3 and 3.4] we have following two lemmas.

Lemma 2.17. *Let $\sigma \geq 1/2$ and $T > 0$. Then for any fixed real numbers t', t'', and σ' we have*

$$\int_{|t| \leq T} \left| \frac{\zeta^4(\sigma + i(t+t'))}{\zeta(1+2it)} \right| \frac{dt}{|\sigma' + i(t+t'')|} \ll (\log T)^5 (\log\log T).$$

Lemma 2.18. *Let z be a complex number and* $\mathrm{Re}(z) > 0$. *For $\sigma \geq 1/2$ we have*

$$\int_{1/2}^{1} \int_{T/2}^{T} \left| \frac{\zeta^4(s+z)}{\zeta(2s)} \right| \left| \frac{x^s}{s} \right| d\sigma dt \ll (\log T)^4 (\log\log T)(x - x^{1/2})(\log x)^{-1}.$$

2.3 Proof of Proposition 2.2

From (2.30) we see that

$$\begin{aligned} C_{k,\beta}(x,y) &= \sum_{n \leqslant y} \left(\sum_{q \leqslant y} c_q^{(\beta)}(n) \right)^k = \prod_{j=1}^{k} \sum_{d_j k_j \leqslant x} d_j^{\beta} \mu(k_j) \sum_{\substack{n \leq y \\ d_j^{\beta} | n}} 1 \\ &= \prod_{j=1}^{k} \sum_{d_j k_j \leqslant x} d_j^{\beta} \mu(k_j) \left\lfloor \frac{y}{[d_1^{\beta}, \ldots, d_k^{\beta}]} \right\rfloor, \end{aligned}$$

where $[d_1^{\beta}, \ldots, d_k^{\beta}]$ denotes the least common multiple of the integers $d_1^{\beta}, d_2^{\beta}, \ldots, d_k^{\beta}$. Let $(d_1^{\beta}, \ldots, d_k^{\beta})$ denote the greatest common divisor of the integers $d_1^{\beta}, d_2^{\beta}, \ldots, d_k^{\beta}$. Then one derives

$$C_{k,\beta}(x,y) = y \prod_{j=1}^{k} \sum_{d_j k_j \leqslant x} (d_1^{\beta}, \ldots, d_k^{\beta}) \mu(k_j) + O\left(\prod_{j=1}^{k} \sum_{d_j k_j \leqslant x} d_j^{\beta} \right) \tag{2.35}$$

for $k \geq 2$. If $k = 1$, then

$$C_{1,\beta}(x,y) = y \sum_{dk \leq x} \mu(k) + O\left(\sum_{dk \leq x} d^{\beta} \right) = y + O\left(\sum_{dk \leq x} d^{\beta} \right).$$

By the aid of the fact that

$$\sum_{n \leqslant x} n^{\beta} = x^{1+\beta} + O(x^{\beta})$$

we deduce that

$$\sum_{dk\leqslant x} d^\beta \mu(k) = \sum_{k\leqslant x}\mu(k)\left(\frac{1}{1+\beta}\Big(\frac{x}{k}\Big)^{1+\beta} + O\left(\frac{x^\beta}{k^\beta}\right)\right) = \frac{x^{1+\beta}}{1+\beta}\sum_{k\leqslant x}\frac{\mu(k)}{k^{1+\beta}} + O\Big(x^\beta \sum_{k\leqslant x}\frac{1}{k^\beta}\Big)$$
$$= \frac{x^{1+\beta}}{(1+\beta)\zeta(1+\beta)} + O(x^\beta \log^{\lfloor 1/\beta\rfloor} x). \tag{2.36}$$

Let d be the greatest common divisor of $d_1,\dots,d_k$. The first sum on the right-hand side of (2.35) can be written as

$$y\sum_{d\le x} d^\beta \prod_{j=1}^{k} \sum_{\substack{l_jk_j\le x/d\\ (l_1,\dots,l_k)=1}} \mu(k_j) = y\sum_{d\le x} d^\beta \prod_{j=1}^{k}\sum_{l_jk_j\le x/d}\mu(k_j)\sum_{l|(l_1,\dots,l_k)}\mu(l)$$
$$= y\sum_{dl\le x} d^\beta\mu(l)\Big(\sum_{mn\le x/dl}\mu(n)\Big)^k$$
$$= \frac{yx^{1+\beta}}{(1+\beta)\zeta(1+\beta)} + O(yx^\beta\log^{\lfloor 1/\beta\rfloor} x), \tag{2.37}$$

where in the last step we used (2.36). The last term on the right-hand side of (2.35) can be estimated as

$$\prod_{j=1}^{k}\sum_{d_jk_j\leqslant x} d_j^\beta \ll \Big(\sum_{dk\leqslant x} d^\beta\Big)^k \ll_k x^{(1+\beta)k}\log^k x.$$

This ends the proof of the proposition.

2.4 Proof of Theorem 2.6

If $\beta = 1$ and $z=0$, the study of the error term in this asymptotic formula is the well-known Dirichlet divisor problem. Thus, we exclude this case. Let $z = a+ib$, $a\le 0$, $b\in\mathbb{R}$, and $c = 1+1/\log x$. Then by Lemma 2.16 we write

$$\sum_{n\le x}\sigma_{z,\beta}(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta(s)\zeta(\beta(s-z))\frac{x^s}{s}ds + E(z,\beta;x),$$

where

$$E(z,\beta;x) \ll \sum_{\substack{x/2<n<2x\\ n\ne x}} \sigma_{a,\beta}(n)\min\left(1,\frac{x}{T|x-n|}\right) + \frac{4^c+x^c}{T}\sum_{n=1}^{\infty}\frac{\sigma_{a,\beta}(n)}{n^c}. \tag{2.38}$$

Form (2.21) one has

$$\sigma_{a,\beta}(n) \le \begin{cases} n^{a\beta}\sigma_{0,\beta}(n), & \text{if } a>0,\\ \sigma_{0,\beta}(n), & \text{if } a\le 0.\end{cases} \tag{2.39}$$

Also we note that

$$\sigma_{0,\beta}(p) = \begin{cases} 2, & \text{if } \beta=1,\\ 1, & \text{if } \beta\ge 2.\end{cases} \tag{2.40}$$

We choose $T = x^{2/3}$. If $0 < |x-n| \le x^{1/3}$, then from (2.39), (2.40), Lemmas 2.14 and 2.15 we have

$$\sum_{0<|x-n|<x^{1/3}} \sigma_{a,\beta}(n) \min\left(1, \frac{x}{T|x-n|}\right) \ll \sum_{0<|x-n|<x^{1/3}} \sigma_{0,\beta}(n) \ll x^{1/3}\log x. \tag{2.41}$$

For $x + x^{1/3} < n < 2x$ one has

$$\begin{aligned} \sum_{x+x^{1/3}<n<2x} \sigma_{a,\beta}(n) \min\left(1, \frac{x}{T|x-n|}\right) &\ll \frac{x}{T} \sum_{x+x^{1/3}<n<2x} \frac{\sigma_{0,\beta}(n)}{n-x} \\ &\ll \frac{x}{T} \sum_{l \ll \log x} \frac{1}{U} \sum_{\substack{U<n-x<2U \\ U=2^l x^{1/3}}} \sigma_{0,\beta}(n). \end{aligned} \tag{2.42}$$

Now by use of Lemmas 2.14 and 2.15 we deduce that

$$\sum_{x+x^{1/3}<n<2x} \sigma_{a,\beta}(n) \min\left(1, \frac{x}{T|x-n|}\right) \ll \frac{x}{T}\log^2 x. \tag{2.43}$$

The same bound holds when $x/2 < n < x - x^{1/3}$. Since

$$\zeta(\sigma) \sim \frac{1}{\sigma-1} \tag{2.44}$$

when $\sigma \to 1+$, then from Lemma 2.11 we find that

$$\frac{4^c + x^c}{T} \sum_{n=1}^{\infty} \frac{\sigma_{a,\beta}(n)}{n^c} \ll \frac{x}{T}\log^2 x. \tag{2.45}$$

Hence from (2.41), (2.43), and (2.45) we deduce that

$$E(z,\beta;x) \ll x^{1/3}\log^2 x.$$

Now we take the integral around the rectangle $\mathcal{D} = [-\alpha - iT, c - iT, c + iT, -\alpha + iT]$, where $\alpha = -a + 1/\log x$. By the residue theorem one writes

$$\frac{1}{2\pi i}\oint_{\mathcal{D}} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s}ds = R,$$

where R is the sum of the residues at the simple poles $s = 1$, $s = z + 1/\beta$, and $s = 0$. The functional equation of $\zeta(s)$ is

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s) := \chi(s)\zeta(1-s). \tag{2.46}$$

From Stirling's formula for the gamma function (A.6) one has

$$\chi(\sigma + it) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)}\left(1 + O\left(\frac{1}{t}\right)\right) \tag{2.47}$$

for fixed σ and $t \geq t_0 > 0$. Hence by (2.46) we have

$$\zeta(s) \ll |t|^{\frac{1}{2}-\sigma} \tag{2.48}$$

for $\sigma < 0$. Also we recall the bound $\zeta(1+it) \ll \log t/\log\log t$ from [Tit86, Theorem 5.16]. Therefore from (2.46) and (2.47) we have $\zeta(it) \ll \sqrt{t}\log t/\log\log t$ for $t \geq 2$. Then by the Phragmén-Lindelöf principle [Tit68, p. 176] one obtains

$$\zeta(\sigma+it) \ll t^{\frac{1}{2}(1-\sigma)}\log t/\log\log t \tag{2.49}$$

for $0 \leq \sigma \leq 1$ and $t \geq 2$. For the upper horizontal integral we have

$$\begin{aligned}
&\int_{-a+iT}^{c+iT} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s}ds \\
&\quad = \frac{1}{T}\left(\int_{-\alpha}^{-\frac{1}{2\log x}} + \int_{-\frac{1}{2\log x}}^{a+\frac{1}{\beta}+\frac{1}{2\log x}} + \int_{a+\frac{1}{\beta}+\frac{1}{2\log x}}^{c}\right)\zeta(\sigma+iT)\zeta(\beta(\sigma+iT-z))x^\sigma d\sigma \\
&\quad = I_1 + I_2 + I_3.
\end{aligned}$$

Using the bound (2.48) for $\zeta(s)$ and (2.49) for $\zeta(\beta(s-z))$ we find

$$I_1 \ll \frac{1}{T}\int_{-\alpha}^{-\frac{1}{2\log x}} T^{\frac{1}{2}-\sigma}T^{\frac{1}{2}(1-\beta\sigma+\beta a)}\log T x^\sigma d\sigma \ll x^{a/3}.$$

Using the bound (2.49) we have

$$I_2 \ll \frac{1}{T}\int_{-\frac{1}{2\log x}}^{a+\frac{1}{\beta}+\frac{1}{2\log x}} T^{\frac{1}{2}(1-\sigma)}T^{\frac{1}{2}(1-\beta\sigma+\beta a)}\log^2 T x^\sigma d\sigma \ll x^{\frac{2-\beta+2a\beta}{3\beta}}\log x.$$

Similarly

$$I_3 \ll \frac{\log^2 T}{T}\int_{a+\frac{1}{\beta}+\frac{1}{2\log x}}^{c} T^{\frac{1}{2}(1-\sigma)}x^\sigma d\sigma \ll x^{1/3}\log x.$$

Therefore we obtain

$$\int_{-a+iT}^{c+iT} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s}ds \ll x^{1/3}\log x.$$

A similar estimate holds for the lower horizontal line. Next we will bound the left vertical part, which is

$$\begin{aligned}
\int_{-\alpha-iT}^{-\alpha+iT} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s}ds &= \int_{-\alpha-iT}^{-\alpha+iT} \chi(s)\chi(\beta(s-z))\zeta(1-s)\zeta(\beta(1-s-z))\frac{x^s}{s}ds \\
&= \sum_{n=1}^{\infty}\frac{\sigma_{z,\beta}(n)}{n}\int_{-\alpha-iT}^{-\alpha+iT}\chi(s)\chi(\beta(s-z))\frac{(xn)^s}{s}ds \\
&= ix^{-\alpha}\sum_{n=1}^{\infty}\frac{\sigma_{z,\beta}(n)}{n^{1+\alpha}}\int_{-T}^{T}\frac{\chi(-\alpha+it)\chi(\beta(-\alpha+it-z))}{(-\alpha+it)(nx)^{-it}}dt,
\end{aligned} \tag{2.50}$$

and where in the first step we used the functional equation (2.46). Note that

$$\frac{1}{-\alpha+it} = \frac{1}{it} + O\left(\frac{1}{t^2}\right). \tag{2.51}$$

Applying (2.47) and (2.51) yields

$$\begin{aligned}\int_{2b+1}^{T} &\frac{\chi(-\alpha+it)\chi(\beta(-\alpha+it-z))}{-\alpha+it}(nx)^{it}dt \\ &\ll \int_{2b+1}^{T} e^{it(\log(nx)+\log(2\pi e)-\log t)+\beta(\log(2\pi e)-\log\beta(t-b))}t^{\alpha+\beta\alpha+\beta a}dt.\end{aligned}$$

Clearly $\alpha+\beta\alpha+\beta a>0$. Let $F(t) := t(\log(nx)+\log(2\pi e)-\log t)+\beta(\log(2\pi e)-\log\beta(t-b))$. Then

$$F''(t) = -\frac{1}{t} - \frac{(t-2b)\beta}{(t-b)^2} < -\frac{1}{T} - \frac{\beta}{T^2}.$$

Therefore by the aid of Lemma 2.13 we deduce that

$$\int_{2b+1}^{T} \frac{\chi(-\alpha+it)\chi(\beta(-\alpha+it-z))}{-\alpha+it}(nx)^{it}dt \ll \frac{T^{\alpha+\beta\alpha+\beta a+1}}{\sqrt{T-\beta}}.$$

Combining this with (2.50) we finally get that

$$\int_{-\alpha-iT}^{-\alpha+iT} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s}ds \ll x^{\frac{1+a}{3}}\log^2 x.$$

Now we compute the residues

$$\operatorname*{res}_{s=0} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s} = -\frac{1}{2}\zeta(-\beta z), \tag{2.52}$$

$$\operatorname*{res}_{s=1} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s} = \zeta(\beta(1-z))x, \tag{2.53}$$

and

$$\operatorname*{res}_{s=\beta^{-1}+z} \zeta(s)\zeta(\beta(s-z))\frac{x^s}{s} = \frac{\zeta(\beta^{-1}+z)}{1+\beta z}x^{\beta^{-1}+z}. \tag{2.54}$$

Clearly the residue in (2.52) is a constant. When $\beta \geq 3$, the residue in (2.54) is absorbed by the error term. This completes the proof of the theorem.

2.5 Proof of Theorem 2.7

Let $z_1 = a_1+ib_1$, $z_2 = a_2+ib_2$, $a_1 \leq 0$, $a_2 \leq 0$, $a_1+a_2 > -1$, $|a_1-a_2| < 1/\beta$ and $b_1, b_2 \in \mathbb{R}$. Define

$$f(z_1,z_2,s;\beta) := \frac{\zeta(s)\zeta(\beta(s-z_1))\zeta(\beta(s-z_2))\zeta(\beta(s-z_1-z_2))}{\zeta(\beta(2s-z_1-z_2))}$$

and let $c = 1 + 1/\log x$. Then in the view of Lemmas 2.12 and 2.16 we may write

$$\sum_{n\leqslant x}\sigma_{z_1,\beta}(n)\sigma_{z_2,\beta}(n) = \frac{1}{2\pi i}\int_{c-iT_0}^{c+iT_0} f(z_1,z_2,s;\beta)\frac{x^s}{s}ds + E(z_1,z_2,\beta;x), \tag{2.55}$$

where

$$\begin{aligned} E(z_1,z_2,\beta;x) \ll &\sum_{\substack{x/2<n<2x\\ n\neq x}}\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)\min\left(1,\frac{x}{T_0|x-n|}\right)\\ &+\frac{4^c+x^c}{T_0}\sum_{n=1}^{\infty}\frac{\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)}{n^c}. \end{aligned}\tag{2.56}$$

Let $T = 2x^{2/3}$ and $T/2 < T_0 < T$. Now we estimate the right-hand side of (2.56). We consider $x + x^{1/3} < n < 2x$. Applying (2.39) to the first term of the right-hand side of (2.56) we obtain

$$\begin{aligned} \sum_{x^{1/3}<n-x<x}\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)\min\left(1,\frac{x}{T_0|x-n|}\right) &= \frac{x}{T_0}\sum_{x^{1/3}<n-x<x}\frac{\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)}{n-x}\\ &\leq \frac{x}{T_0}\sum_{0\leq l\ll\log x}\sum_{\substack{U<n-x<2U\\ U=2^l x^{1/3}}}\frac{(\sigma_{0,\beta}(n))^2}{n-x}\\ &\leq \frac{x}{T_0}\sum_{0\leq l\ll\log x}\frac{1}{U}\sum_{\substack{x+U<n<x+2U\\ U=2^l x^{1/3}}}(\sigma_{0,\beta}(n))^2. \end{aligned}\tag{2.57}$$

From (2.40) and Lemmas 2.14 and 2.15 we deduce

$$\sum_{x+U<n<x+2U}(\sigma_{0,\beta}(n))^2 \ll \frac{U}{\log x}\exp(4\log\log x) \ll U\log^3 x. \tag{2.58}$$

Invoking this in (2.57) we finally have

$$\sum_{x^{1/3}<n-x<x}\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)\min\left(1,\frac{x}{T_0|x-n|}\right) \ll \frac{x}{T_0}\log^4 x. \tag{2.59}$$

Similarly

$$\sum_{x/2<n<x-x^{1/3}}\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)\min\left(1,\frac{x}{T_0|x-n|}\right) \ll \frac{x}{T_0}\log^4 x. \tag{2.60}$$

Let $x - x^{1/3} \leq n \leq x + x^{1/3}$. Using (2.39) and (2.82) one has

$$\begin{aligned} \sum_{0<|x-n|\leq x^{1/3}}\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)\min\left(1,\frac{x}{T_0|x-n|}\right) &\leq \sum_{0<|x-n|\leq x^{1/3}}\sigma_{0,\beta}(n)\sigma_{0,\beta}(n)\\ &\ll x^{1/3}\log^3 x. \end{aligned}\tag{2.61}$$

From (2.44) and Lemma 2.12 we have

$$\frac{4^c + x^c}{T_0} \sum_{n=1}^{\infty} \frac{\sigma_{a_1,\beta}(n)\sigma_{a_2,\beta}(n)}{n^c} \ll \frac{x}{T_0} f(a_1, a_2, c; \beta) \ll \frac{x}{T_0} \log^4 x, \tag{2.62}$$

for $\beta \geq 1$. Combining (2.59), (2.60), (2.61) and (2.62) we obtain

$$E(z_1, z_2, \beta; x) \ll x^{1/3} \log^4 x. \tag{2.63}$$

Let $\lambda = \frac{1}{2}(a_1 + a_2 + 1/\beta)$. Suppose $\mathcal{R}$ is a positively oriented contour with vertices $c \pm iT_0$ and $\lambda \pm iT_0$. By residue theorem we have

$$\frac{1}{2\pi i} \oint_{\mathcal{R}} f(z_1, z_2, s; \beta) \frac{x^s}{s} ds = R_0, \tag{2.64}$$

where R_0 is the sum of residues in side the contour $\mathcal{R}$. By Hölder's inequality [Tit68, p. 382] one has

$$\begin{aligned}
&\left(\int_{\lambda - iT_0}^{\lambda + iT_0} f(z_1, z_2, s; \beta) \frac{x^s}{s} ds \right)^4 \\
&\ll \int_{-T_0}^{T_0} \frac{|\zeta(\lambda + it)|^4 x^\lambda dt}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} \int_{-T_0}^{T_0} \frac{|\zeta(\beta(\lambda + it - z_1))|^4 x^\lambda dt}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} \\
&\quad \times \int_{-T_0}^{T_0} \frac{|\zeta(\beta(\lambda + it - z_2))|^4 x^\lambda dt}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} \int_{-T_0}^{T_0} \frac{|\zeta(\beta(\lambda + it - z_1 - z_2))|^4 x^\lambda dt}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|}.
\end{aligned} \tag{2.65}$$

By Lemma 2.17 we find that

$$\int_{-T_0}^{T_0} \frac{|\zeta(\beta(\lambda + it - z_1 - z_2))|^4 x^\lambda}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \ll x^\lambda (\log^5 T_0) \log\log T_0$$

Let $a_1 - a_2 \geq 0$. Then by Lemma 2.17 we have

$$\int_{-T_0}^{T_0} \frac{|\zeta(\beta(\lambda + it - z_2))|^4 x^\lambda}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \ll x^\lambda (\log^5 T_0) \log\log T_0.$$

By the functional equation (2.46) and (2.47) one obtains

$$\begin{aligned}
&\int_{-T_0}^{T_0} \frac{|\zeta(\beta(\lambda + it - z_1))|^4}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \\
&\quad = \int_{-T_0}^{T_0} |\chi(\beta(\lambda + it - z_1))| \frac{|\zeta(1 - \beta(\lambda + it - z_1))|^4}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \\
&\quad \ll T_0^{(a_1 - a_2)\beta/2} \int_{-T_0}^{T_0} \frac{|\zeta(1/2 + (a_1 - a_2)\beta/2 + i(t + t'))|^4}{|\zeta(1 + 2it))(\lambda + i(t + t''))|} dt,
\end{aligned}$$

where in last step we made a suitable change of variable. Finally, by Lemma 2.17 we obtain

$$\int_{-T_0}^{T_0} \frac{\zeta(\beta(\lambda + it - z_1))|^4 x^\lambda}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \ll x^\lambda T_0^{(a_1 - a_2)\beta/2} (\log^5 T_0) \log\log T_0.$$

The case $a_1 - a_2 \leq 0$ can be treated similarly. Therefore for any sign of $a_1 - a_2$ we have

$$\int_{-T_0}^{T_0} \frac{\zeta(\beta(\lambda + it - z_i))|^4 x^\lambda}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \ll x^\lambda T_0^{|a_1 - a_2|\beta/2} (\log^5 T_0) \log\log T_0,$$

for $i = 1, 2$. Using a similar argument one can deduce that

$$\int_{-T_0}^{T_0} \frac{|\zeta(\lambda + it)|^4 x^\lambda}{|\zeta(\beta(2(\lambda + it) - z_1 - z_2))(\lambda + it)|} dt \ll x^\lambda T_0^{1/2 - \lambda} (\log^5 T_0) \log\log T_0.$$

Thus from (2.65) we have

$$\int_{\lambda - iT_0}^{\lambda + iT_0} f(z_1, z_2, s; \beta) \frac{x^s}{s} ds \ll \max\left(x^{\frac{1}{2\beta} + \frac{a_1 + a_2}{2} + \frac{\beta|a_1 - a_2|}{3}}, x^{\frac{1}{3} + \frac{1}{6\beta} + \frac{a_1 + a_2}{6}} \right) \times (\log^5 x) \log\log x. \tag{2.66}$$

Next we compute the double integral

$$\int_\lambda^c \int_{T/2}^T f(z_1, z_2, \sigma + it; \beta) \frac{x^{\sigma + it}}{\sigma + it} d\sigma dt.$$

Using the functional equation (2.46) we write

$$\begin{aligned}
&\int_\lambda^c \int_{T/2}^T \frac{|\zeta(\sigma + it)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt \\
&= \int_\lambda^{1/2} \int_{T/2}^T \frac{|\chi(\sigma + it)||\zeta(1 - \sigma - it)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt \\
&\quad + \int_{1/2}^c \int_{T/2}^T \frac{|\zeta(\sigma + it)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt.
\end{aligned}$$

Now by the aid of (2.47) and Lemma 2.18 we deduce that

$$\int_\lambda^c \int_{T/2}^T \frac{|\zeta(\sigma + it)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt \ll \frac{x}{\log x} (\log^4 T) \log\log T.$$

If $a_1 - a_2 \geq 0$, then similarly we can find that

$$\int_\lambda^c \int_{T/2}^T \frac{|\zeta(\beta(\sigma + it - z_1)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt \ll \frac{x}{\log x} (\log^4 T) \log\log T.$$

From Lemma 2.18 we have

$$\int_\lambda^c \int_{T/2}^T \frac{|\zeta(\beta(\sigma + it - z_2)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt \ll \frac{x}{\log x} (\log^4 T) \log\log T$$

and

$$\int_\lambda^c \int_{T/2}^T \frac{|\zeta(\beta(\sigma + it - z_1 - z_2)|^4 x^\sigma}{|\zeta(\beta(2(\sigma + it) - z_1 - z_2))(\sigma + it)|} d\sigma dt \ll \frac{x}{\log x} (\log^4 T) \log\log T.$$

Therefore by Hölder's inequality

$$\int_{\lambda}^{c}\int_{T/2}^{T} f(z_1,z_2,\sigma+it;\beta)\frac{x^{\sigma+it}}{\sigma+it}d\sigma dt \ll \frac{x}{\log x}(\log^4 T)\log\log T.$$

Hence we can choose a suitable T_0 so that $T/2 \leq T_0 \leq T$ and

$$\begin{aligned}\int_{\lambda}^{c} f(z_1,z_2,\sigma+iT_0;\beta)\frac{x^{\sigma+iT_0}}{\sigma+iT_0}d\sigma &\ll \frac{x}{T_0\log x}(\log^4 T_0)\log\log T_0\\ &\ll x^{1/3}(\log^3 x)\log\log x. \end{aligned} \tag{2.67}$$

Finally combining (2.55), (2.63), (2.64), (2.66) and (2.67) we find

$$\begin{aligned}\sum_{n\leqslant x}\sigma_{z_1,\beta}(n)\sigma_{z_2,\beta}(n) = R_0 \\ + O\Bigg(\max\left(x^{\frac{1}{2\beta}+\frac{a_1+a_2}{2}+\frac{\beta|a_1-a_2|}{3}}, x^{\frac{1}{3}+\frac{1}{6\beta}+\frac{a_1+a_2}{6}}\right)(\log^5 x)\log\log x\Bigg).\end{aligned} \tag{2.68}$$

Since $z_i \neq 0$, $z_1 \neq z_2$, and $|\operatorname{Re}(z_1-z_2)| < 1/\beta$, then all the poles are simple. The residues at the simple poles $s=1$, $s=z_1+1/\beta$, $s=z_2+1/\beta$, and $s=z_1+z_2+1/\beta$ are given by

$$\operatorname*{res}_{s=1} f(z_1,z_2,s;\beta)\frac{x^s}{s} = \frac{\zeta(\beta(1-z_1))\zeta(\beta(1-z_2))\zeta(\beta(1-z_1-z_2))}{\zeta(\beta(2-z_1-z_2))}x, \tag{2.69}$$

$$\operatorname*{res}_{s=z_1+1/\beta} f(z_1,z_2,s;\beta)\frac{x^s}{s} = \frac{\zeta(z_1+\frac{1}{\beta})\zeta(1+\beta z_1-\beta z_2)\zeta(1-\beta z_2)}{(z_1\beta+1)\zeta(2+\beta z_1-\beta z_2)}x^{z_1+\frac{1}{\beta}}, \tag{2.70}$$

$$\operatorname*{res}_{s=z_2+1/\beta} f(z_1,z_2,s;\beta)\frac{x^s}{s} = \frac{\zeta(z_2+\frac{1}{\beta})\zeta(1+\beta z_2-\beta z_1)\zeta(1-\beta z_1)}{(\beta z_2+1)\zeta(2+\beta z_2-\beta z_1)}x^{z_2+\frac{1}{\beta}}, \tag{2.71}$$

and

$$\operatorname*{res}_{s=z_1+z_2+1/\beta} f(z_1,z_2,s;\beta)\frac{x^s}{s} = \frac{\zeta(z_1+z_2+\frac{1}{\beta})\zeta(\beta z_2+1)\zeta(\beta z_1+1)}{(\beta z_1+\beta z_2+1)\zeta(2+\beta z_1+\beta z_2)}x^{z_1+z_2+\frac{1}{\beta}}. \tag{2.72}$$

If $\beta \geq 3$, then (2.70), (2.71), and (2.72) are smaller than the error term of the right-hand side of (2.68). Therefore for $\beta \geq 3$ we find

$$R_0 = \frac{\zeta(\beta(1-z_1))\zeta(\beta(1-z_2))\zeta(\beta(1-z_1-z_2))}{\zeta(\beta(2-z_1-z_2))}x.$$

For $\beta = 1$ and $-1/2 < \operatorname{Re}(z_1), \operatorname{Re}(z_2), \operatorname{Re}(z_1+z_2) < 0$ we find

$$\begin{aligned}R_0 =& x\Bigg(\frac{\zeta(1-z_1)\zeta(1-z_2)\zeta(1-z_1-z_2)}{\zeta(2-z_1-z_2)} + \frac{\zeta(z_1+1)\zeta(1+z_1-z_2)\zeta(1-z_2)}{(z_1+1)\zeta(2+z_1-z_2)}x^{z_1}\\ &+ \frac{\zeta(z_2+1)\zeta(1+z_2-z_1)\zeta(1-z_1)}{(z_2+1)\zeta(2+z_2-z_1)}x^{z_2} + \frac{\zeta(z_1+z_2+1)\zeta(z_2+1)\zeta(z_1+1)}{(z_1+z_2+1)\zeta(2+z_1+z_2)}x^{z_1+z_2}\Bigg).\end{aligned}$$

Finally for $\beta = 2$ and $-1/10 < \operatorname{Re}(z_1), \operatorname{Re}(z_2), \operatorname{Re}(z_1+z_2) < 0$ we obtain

$$R_0 = \sqrt{x}\Bigg(\frac{\zeta(2(1-z_1))\zeta(2(1-z_2))\zeta(2(1-z_1-z_2))}{\zeta(2(2-z_1-z_2))}\sqrt{x}$$

$$+\frac{\zeta(z_1+\frac{1}{2})\zeta(1+2z_1-2z_2)\zeta(1-2z_2)}{(2z_1+1)\zeta(2+2z_1-2z_2)}x^{z_1}$$
$$+\frac{\zeta(z_2+\frac{1}{2})\zeta(1+2z_2-2z_1)\zeta(1-2z_1)}{(2z_2+1)\zeta(2+2z_2-2z_1)}x^{z_2}$$
$$+\qquad\frac{\zeta(z_1+z_2+\frac{1}{2})\zeta(2z_2+1)\zeta(2z_1+1)}{(2z_1+2z_2+1)\zeta(2+2z_1+2z_2)}x^{z_1+z_2}\Bigg).$$

This completes the proof of the theorem.

2.6 Proof of Theorem 2.3

Let us consider

$$\alpha = 1+\frac{1}{\log y},$$

as well as $y \geq x$ and $T = y^{2/3}$. By Lemma 2.16 one finds

$$\sum_{q^\beta \leq x} c_{q,\beta}(n) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\sigma_{1-s,\beta}(n)}{\zeta(\beta s)}\frac{x^s}{s}ds + E_1(x,n), \tag{2.73}$$

where

$$E_1(x,n) \ll \sum_{\substack{x/2<q^\beta<2x\\ q^\beta\neq x}} |c_{q,\beta}(n)|\min\left(1,\frac{x}{T|x-q^\beta|}\right)+\frac{x^\alpha}{T}\sum_{q=1}^{\infty}\frac{|c_{q,\beta}(n)|}{q^{\beta\alpha}}. \tag{2.74}$$

Using Lemma 2.10, we have

$$\sum_{q=1}^{\infty}\frac{|c_{q,\beta}(n)|}{q^{\beta\alpha}} = \sum_{q=1}^{\infty}\frac{1}{q^{\beta\alpha}}\left|\sum_{\substack{d|q\\ d^\beta|n}} d^\beta\mu\left(\frac{q}{d}\right)\right| \leq \sum_{q=1}^{\infty}\frac{1}{q^{\beta\alpha}}\sum_{d^\beta|n} d^{\beta-\beta\alpha} = \sigma_{-\frac{1}{\log y},\beta}(n)\zeta(\beta\alpha).$$

Therefore from (2.44) we deduce

$$\sum_{q=1}^{\infty}\frac{|c_{q,\beta}(n)|}{q^{\alpha}} \ll \begin{cases}\sigma_{-\frac{1}{\log y},\beta}(n)\log y, & \text{if } \beta>1,\\ \sigma_{-\frac{1}{\log y},\beta}(n), & \text{if } \beta=1.\end{cases} \tag{2.75}$$

Similarly

$$\sum_{\substack{x/2<q^\beta<2x\\ q^\beta\neq x}} |c_{q,\beta}(n)|\min\left(1,\frac{x}{T|x-q^\beta|}\right) \leq \sum_{d^\beta|n} d^\beta \sum_{\substack{x/2<d^\beta q^\beta<2x\\ d^b q^\beta\neq x}} \min\left(1,\frac{x}{T|x-d^\beta q^\beta|}\right)$$
$$\ll \frac{x}{T}\sigma_{0,\beta}(n)\log x. \tag{2.76}$$

Hence from (2.74), (2.75) and (2.76) we obtain

$$E_1(x,n) \ll \frac{x}{T}\sigma_{0,\beta}(n)\log y. \tag{2.77}$$

Now by (2.40) and Lemma 2.14 one has

$$\sum_{n\le y} E_1(x,n) \ll \begin{cases} \frac{xy}{T}\log y, & \text{if } \beta>1,\\ \frac{xy}{T}\log^2 y, & \text{if } \beta=1. \end{cases} \tag{2.78}$$

Summing the both sides of (2.73) over n and using Theorem 2.6 we can write

$$\begin{aligned} C_{1,\beta}(x,y) = \sum_{n\leqslant y}\sum_{q^\beta\leqslant x} c_{q,\beta}(n) &= \frac{y}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{x^s}{s}ds \\ &+ \frac{y^{1+1/\beta}}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}\frac{\zeta(1-s+1/\beta)}{1+\beta-\beta s}\frac{(x/y)^s}{s\zeta(\beta s)}ds \\ &+ O\left(y^{1/3}\log^2 y\int_2^T\frac{x^\alpha}{t|\zeta(\beta(\alpha+it))|}dt\right) + \sum_{n\le y}E_1(x,n). \end{aligned} \tag{2.79}$$

Note that $(\zeta(\sigma+it))^{\pm1} \ll \log t$ for $1\le\sigma\le 1/\log y$. Thus the third term in the right-hand side of (2.79) is

$$\ll xy^{1/3}\log^2 y\log^2 T. \tag{2.80}$$

By (2.33) the first integral in the right-hand side of (2.79) is

$$y + O\left(\frac{yx}{T}\right)$$

for $x\ge 2$. For the second integral we shift the line of integration from $\sigma=\alpha$ to $\sigma = 1+\alpha+1/\beta$. The residue due to the simple pole at $s=1+1/\beta$ is

$$\operatorname*{res}_{s=1+1/\beta}\frac{\zeta(1-s+1/\beta)}{1+\beta-\beta s}\frac{y^{1+1/\beta}(x/y)^s}{s\zeta(\beta s)} = \frac{x^{1+1/\beta}}{2(1+\beta)\zeta(1+\beta)}.$$

The contribution from the horizontal line is

$$\begin{aligned} &\ll \frac{y^{1+\frac{1}{\beta}}}{T^2}\left(\log T\int_\alpha^{\alpha+\frac{1}{\beta}} T^{\frac{1}{2}(\sigma-\frac{1}{\beta})}\left(\frac{x}{y}\right)^\sigma d\sigma + \int_{\alpha+\frac{1}{\beta}}^{1+\alpha+\frac{1}{\beta}} T^{-\frac{1}{2}+\sigma-\frac{1}{\beta}}\left(\frac{x}{y}\right)^\sigma d\sigma\right) \\ &\ll \frac{xy^{1/\beta}}{T^{3/2}}\log T + \frac{x^{1+1/\beta}}{T^{1/2}}\log T \end{aligned}$$

Similarly the contribution from right vertical line is

$$\ll \frac{x^{2+1/\beta}}{y^\alpha}\int_2^T t^{-\frac{1}{2}+\frac{1}{\log y}}dt \ll \frac{x^{2+1/\beta}}{y^\alpha}T^{\frac{1}{2}+\frac{1}{\log y}}.$$

Note that the second integral of (2.79) disappears when $\beta\ge 3$. Finally, replace x by x^β to end the proof.

2.7 Proof of Theorem 2.4

For $j \in \{1,2\}$, we let α_j be such that

$$\alpha_j = 1 + \frac{j}{\log y}.$$

Let $y \geq x$ and $T = x^2 \log^5 x$. From (2.73) and (2.77) we have

$$\sum_{q^\beta \leqslant x} c_{q,\beta}(n) = \frac{1}{2\pi i} \int_{\alpha_j - iT}^{\alpha_j + iT} \frac{\sigma_{1-s,\beta}(n)}{\zeta(\beta s)} \frac{x^s}{s} ds + O\bigg(\frac{x}{T} \sigma_{0,\beta}(n) \log y\bigg).$$

Note that

$$\frac{1}{2\pi i} \int_{\alpha_j - iT}^{\alpha_j + iT} \frac{\sigma_{1-s,\beta}(n)}{\zeta(\beta s)} \frac{x^s}{s} ds \ll x \sigma_{0,\beta}(n) \log^2 T$$

Therefore

$$\begin{aligned}\left(\sum_{q^b \leqslant x} c_{q,\beta}(n)\right)^2 &= \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - 2iT}^{\alpha_2 + 2iT} \frac{\sigma_{1-s_1,\beta}(n)}{\zeta(\beta s_1)} \frac{\sigma_{1-s_2,\beta}(n)}{\zeta(\beta s_2)} \frac{x^{s_1+s_2}}{s_1 s_2} ds_2 ds_1 \\ &\quad + O\left(\frac{x^2}{T} (\sigma_{0,\beta}(n))^2 \log y \log^2 T\right). \end{aligned} \tag{2.81}$$

Combining equation (2.40), Lemmas 2.14 and 2.15 we find

$$\sum_{n<y} (\sigma_{0,\beta}(n))^2 \ll y \log^3 y. \tag{2.82}$$

Now sum over n on both sides of (2.81) so that

$$\begin{aligned} C_{2,\beta}(x,y) &= \sum_{n \leqslant y} \left(\sum_{q \leqslant x} c_{q,\beta}(n)\right)^2 \\ &= \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - 2iT}^{\alpha_2 + 2iT} \frac{G(s_1, s_2, \beta, n)}{\zeta(s_1)\zeta(s_2)} \frac{x^{s_1+s_2}}{s_1 s_2} ds_2 ds_1 \\ &\qquad + O\left(\frac{x^2 y}{T} \log^4 y \log^2 T\right) \\ &= I + O\left(\frac{x^2 y}{T} \log^4 y \log^2 T\right), \end{aligned}$$

where

$$G(s_1, s_2, \beta, y) = \sum_{n \leqslant y} \sigma_{1-s_1,\beta}(n) \sigma_{1-s_2,\beta}(n).$$

From Theorem 2.7 we find

$$I = I_1 + I_2 + I_3 + I_4 + O(x^2 y^{\frac{1}{3} + \frac{1}{6\beta}} (\log^5 y)(\log^4 T) \log\log y),$$

where

$$I_1 = \frac{y}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-2iT}^{\alpha_2+2iT}\frac{\zeta(\beta(s_1+s_2-1))}{\zeta(\beta(s_1+s_2))}\frac{x^{s_1+s_2}}{s_1s_2}ds_1ds_2,$$

$$I_2 = \frac{y^{1+1/\beta}}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-2iT}^{\alpha_2+2iT}\frac{\zeta(1-s_1+1/\beta)\zeta(1-\beta s_1+\beta s_2)\zeta(1-\beta+\beta s_2))}{(1+\beta-\beta s_1)\zeta(2-\beta s_1+\beta s_2)\zeta(\beta s_1)\zeta(\beta s_2)} \times \frac{y^{-s_1}x^{s_1+s_2}}{s_1s_2}ds_1ds_2,$$

$$I_3 = \frac{y^{1+1/\beta}}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-2iT}^{\alpha_2+2iT}\frac{\zeta(1-s_2+1/\beta)\zeta(1-\beta s_2+\beta s_1)\zeta(1-\beta+\beta s_1))}{(1+\beta-\beta s_2)\zeta(2-\beta s_2+\beta s_1)\zeta(\beta s_1)\zeta(\beta s_2)} \times \frac{y^{-s_2}x^{s_1+s_2}}{s_1s_2}ds_1ds_2,$$

as well as

$$I_4 = \frac{y^{2+1/\beta}}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\alpha_2-2iT}^{\alpha_2+2iT}\frac{\zeta(2-s_1-s_2+1/\beta)\zeta(1+\beta-\beta s_1)\zeta(1+\beta-\beta s_2))}{(1+2\beta-\beta s_1-\beta s_2)\zeta(2+2\beta-\beta s_2-\beta s_1)\zeta(\beta s_1)\zeta(\beta s_2)} \times \frac{(x/y)^{s_1+s_2}}{s_1s_2}ds_1ds_2.$$

Note that the integrals I_2, I_3, and I_4 disappear when $\beta \geq 3$. First we will compute the integral I_1. Let

$$J_1(s_1) = \frac{1}{2\pi i}\int_{\alpha_2-iT}^{\alpha_2+iT}\frac{\zeta(\beta(s_1+s_2-1))}{\zeta(\beta(s_1+s_2))}\frac{x^{s_2}}{s_2}ds_2.$$

Shift the line of integration from $\sigma = \alpha_2$ to $\sigma = 1+\frac{1}{2\beta}-\alpha_1$. Note that the integrand has a simple pole at $1+1/\beta-s_1$ in this region. The residue is

$$\operatorname*{res}_{s_2=1-s_1+1/\beta}\frac{\zeta(\beta(s_1+s_2-1))}{\zeta(\beta(s_1+s_2))}\frac{x^{s_2}}{s_2} = \frac{x^{1-s_1+1/\beta}}{(1+\beta-\beta s_1)\zeta(1+\beta)}.$$

Let $T \geq x^{2/\beta}$. The contribution from the horizontal line is

$$\ll \frac{1}{T^{\frac{1}{2}}}\int_{1-\alpha_1+1/2\beta}^{1/\beta}\left(\frac{x}{T^{\beta/2}}\right)^{\sigma}d\sigma + \frac{\log x}{T}\int_{1/\beta}^{\alpha_2}(x)^{\sigma}d\sigma \ll \frac{x}{T}+\frac{x^{1/2\beta}}{T^{3/4}}.$$

Hence

$$\begin{aligned} I_1 =& \frac{y}{(2\pi i)^2}\int_{\alpha_1-iT}^{\alpha_1+iT}\int_{\frac{1}{2\beta}-\frac{1}{\log y}-2iT}^{\frac{1}{2\beta}-\frac{1}{\log y}+2iT}\frac{\zeta(\beta(s_1+s_2-1))}{\zeta(\beta(s_1+s_2))}\frac{x^{s_1+s_2}}{s_1s_2}ds_1ds_2 \\ &+ \frac{y}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT}\frac{x^{1+1/\beta}}{s_1(1+\beta-\beta s_1)\zeta(1+\beta)}ds_1 + O\bigg(\frac{yx^2}{T}\log x + \frac{yx^{1+1/2\beta}}{T^{3/4}}\log x\bigg). \end{aligned} \tag{2.83}$$

Denote the first integral in the right-hand side of (2.83) by I_{11}. Then we have

$$\begin{aligned} I_{11} &\ll yx^{1+1/2\beta}\int_1^T\int_1^{2T}\frac{|\zeta(1/2+i\beta(t_1+t_2))|}{t_1t_2}dt_1dt_2 \\ &\ll yx^{1+1/2\beta}\log^2 T\int_{T/2}^{T}\int_T^{2T}\frac{|\zeta(1/2+i\beta(t_1+t_2))|}{t_1t_2}dt_1dt_2 \end{aligned}$$

$$\ll yx^{1+1/2\beta}\log^2 T\bigg(\int_{T/2}^{T}\int_{T}^{2T}|\zeta(1/2+i\beta(t_1+t_2))|^2 dt_1 dt_2 \int_{T/2}^{T}\int_{T}^{2T}\frac{1}{t_1^2 t_2^2}dt_1 dt_2\bigg)^{1/2},$$

where in the last step we used Hölder's inequality. By the aid of the mean value theorem of $\zeta(s)$ [Tit86, Theorem 7.3] we deduce that

$$I_{11} \ll yx^{1+1/2\beta}\log^3 T.$$

Finally, by applying the residue theorem on the second integral in the right-hand side of (2.83) we conclude that

$$I_1 = \frac{yx^{1+1/\beta}}{(1+\beta)\zeta(1+\beta)} + O\bigg(yx^{1+1/2\beta}\log^3 T + \frac{yx^2}{T}\log x + \frac{yx^{1+1/2\beta}}{T^{3/4}}\log x\bigg).$$

Next we compute the integral I_2. Let

$$J_2(s_2) = \frac{1}{2\pi i}\int_{\alpha_1-iT}^{\alpha_1+iT}\frac{\zeta(1-s_1+1/\beta)\zeta(1-\beta s_1+\beta s_2)}{(1+\beta-\beta s_1)\zeta(2-\beta s_1+\beta s_2)\zeta(\beta s_1)}\frac{y^{-s_1}x^{s_1}}{s_1}ds_1$$

Shift the line of integration in the s_1-plane from $\sigma=\alpha_1$ to $\sigma=\alpha_3=1+\frac{1}{\beta}-\frac{3}{\log y}$. Note that $s_1=s_2$ is a simple pole of the integrand and the residue is

$$\operatorname*{res}_{s_1=s_2}\frac{\zeta(1-s_1+1/\beta)\zeta(1-\beta s_1+\beta s_2)}{(1+\beta-\beta s_1)\zeta(2-\beta s_1+\beta s_2)\zeta(\beta s_1)}\frac{y^{-s_1}x^{s_1}}{s_1}$$
$$= -\frac{\zeta(1-s_2+1/\beta)}{\beta\zeta(2)(1+\beta-\beta s_2)\zeta(\beta s_2)}\frac{(x/y)^{s_2}}{s_2}.$$

The contribution from the horizontal lines is

$$\ll \frac{x^{1+1/\beta}}{y^{1+1/\beta}T}\log T.$$

provided $x \le y < x^{\beta+2}\log^{\frac{5}{2}(\beta+1)} y$. The contribution from the vertical line $\sigma=\alpha_3$ is

$$\ll \frac{x^{1+1/\beta}}{y^{1+1/\beta}}\log T.$$

Therefore

$$I_2 = \frac{y^{1+1/\beta}}{\beta\zeta(2)}\frac{1}{2\pi i}\int_{\alpha_2-2iT}^{\alpha_2+2iT}\frac{\zeta(1-s_2+1/\beta)\zeta(1-\beta+\beta s_2)}{(1+\beta-\beta s_2)\zeta^2(\beta s_2)}\frac{(x^2/y)^{s_2}}{s_2^2}ds_2$$
$$+ O\bigg(\frac{x^{2+1/\beta}}{T}\log^2 T + x^{2+1/\beta}\log^2 T\bigg).$$

Next we shift the line of integration in the s_2-plane form $\sigma=\alpha_2$ to $\sigma=\alpha_4=1+\alpha_2+\frac{1}{\beta}$. Note that $s_2=1+\frac{1}{\beta}$ is a simple pole and the residue is

$$\operatorname*{res}_{s_2=1+1/\beta}\frac{y^{1+1/\beta}\zeta(1-s_2+1/\beta)\zeta(1-\beta+\beta s_2)}{\beta\zeta(2)(1+\beta-\beta s_2)\zeta^2(\beta s_2)}\frac{(x^2/y)^{s_2}}{s_2^2} = \frac{x^{2+2/\beta}}{2(1+\beta)^2\zeta^2(1+\beta)}.$$

If we split the interval $(\alpha_2, 1+\alpha_2+1/\beta)$ into two subintervals $(\alpha_2, 1+1/b)$ and $(1+1/b, 1+a_2+1/\beta)$, then the horizontal line integration is

$$\ll (y^{-1}x^2)^{1+1/\beta}T^{-5/2} + (y^{-1}x^2)^{2+1/\beta}T^{-2}.$$

The vertical line integration is

$$\ll y^{-1}x^{4+2/\beta}.$$

To bound the integral I_3 we move the line of integration in s_2 plane from $\sigma = \alpha_2$ to $\sigma = \alpha_2 + \frac{1}{\beta}$. The contribution from the horizontal line is

$$\ll \frac{xy^{1+1/\beta}}{T^2}\log^2 T \int_{\alpha_2}^{a_2+1/\beta} T^{\frac{1}{2}(\beta\sigma+\sigma-1/\beta)}(x/y)^{\sigma}d\sigma \ll \frac{x^{2+1/\beta}}{T}\log^2 T,$$

provided that $x \le y < x^{\beta+2}\log^{\frac{5}{2}(\beta+1)} y$. If $\alpha_5 = \alpha_2 + 1/\beta$, then the contribution from the left vertical line is

$$\ll x^{2+1/\beta}\int_2^T\int_2^T \frac{\sqrt{t_2(t_2-t_1)}}{t_2^2 t_1}dt_2dt_1 \ll x^{2+1/\beta}\log^2 T.$$

Similarly if one moves the line of integration in s_2-plane from $\sigma = \alpha_2$ to $\sigma = 1+\frac{1}{\beta}-\frac{4}{\log x}$, then it can be shown that

$$I_4 \ll x^{2+1/\beta}\log^2 T.$$

Now we complete the proof of the theorem by replacing x by x^{β}.

2.8 Proof of Theorem 2.5

To prove this, we recall the following definitions [GK91, pp. 30-31] as well as auxiliary lemma from [GK91, Lemma 4.3].

Definition 2.19. Let N, y, s and ε be positive real numbers with $\varepsilon < \frac{1}{2}$, and let P be a non-negative integer. Define $\mathbf{F}(N, P, s, y, \varepsilon)$ to be the set of functions f such that:

(1) f is defined and has continuous P derivatives on some interval $[a, b]$, with $[a, b] \subseteq [N, 2N]$,

(2) if $0 \le p \le P-1$ and $a \le x \le b$ then

$$|f^{(p+1)}(x) - (-1)^p(s)_p yx^{-s-p}| < \varepsilon(s)_p yx^{-s-p},$$

where $(\alpha)_0 = 1$ and $(\alpha)_{n+1} = (\alpha+n)(\alpha)_n$.

Definition 2.20. Let $\mathbf{I} = (a, b]$ where a and b and integers. Let k and l be real numbers such that $0 \le k \le 1/2 \le l \le 1$. Suppose that for every $s > 0$, there is some $P = P(k, l, s)$ and some $\varepsilon = \varepsilon(k, l, s) < 1/2$ such that for every $N > 0$, every $y > 0$, and every $f \in \mathbf{F}(N, P, s, y, \varepsilon)$, the estimate

$$\sum_{n\in\mathbf{I}} e(f(n)) \ll (yN^{-s})^k N^l + y^{-1}N^s$$

holds. Here it is also assumed that f is defined on $[a, b]$ and the implied constant depends only on k, l and s. We then say that (k, l) is an exponent pair.

For real t, $[t]$ denotes the integral part of t and

$$\psi(t) := t - [t] - \tfrac{1}{2} \tag{2.84}$$

denotes the saw-tooth function.

Definition 2.21. Let N, y, s and ε be positive real numbers with $\varepsilon < \frac{1}{2}$, and let P be a non-negative integer. Define $\mathbf{F}(N, P, s, y, \varepsilon)$ to be the set of functions f such that:

(1) f is defined and has continuous P derivatives on some interval $[a, b]$, with $[a, b] \subset [N, 2N]$,

(2) if $0 \leq p \leq P - 1$ and $a \leq x \leq b$ then

$$|f^{(p+1)}(x) - (-1)^p (s)_p y x^{-s-p}| < \varepsilon (s)_p y x^{-s-p},$$

where $(\alpha)_0 = 1$ and $(\alpha)_{n+1} = (\alpha + n)(\alpha)_n$.

Lemma 2.22. *Suppose that (k, l) is an exponent pair, $\mathbf{I}$ is a subinterval of $(N, 2N]$, then*

$$\sum_{n \in \mathbf{I}} \psi(y/n) \ll y^{k/(k+1)} N^{(l-k)/(k+1)} + y^{-1} N^2.$$

By the use of (2.30) we have

$$C_{1,\beta}(x, y) = \sum_{n \leqslant y} \sum_{q \leqslant x} c_q^{(\beta)}(n) = \sum_{n \leqslant y} \sum_{q \leqslant x} \sum_{\substack{d \mid q \\ d^\beta \mid n}} d^\beta \mu\left(\frac{q}{d}\right) = \sum_{n \leqslant y} \sum_{\substack{dk \leqslant x \\ d^\beta \mid n}} d^\beta \mu(k)$$

where we have made the change $k = \frac{q}{d}$. Interchanging the order of summation we obtain

$$\begin{aligned} C_{1,\beta}(x, y) &= \sum_{dk \leqslant x} d^\beta \mu(k) \sum_{\substack{n \leq y \\ d^\beta \mid n}} 1 = \sum_{dk \leqslant x} d^\beta \mu(k) \left[\frac{y}{d^\beta}\right] \\ &= y \sum_{dk \leqslant x} \mu(k) - \frac{1}{2} \sum_{dk \leqslant x} d^\beta \mu(k) - \sum_{dk \leqslant x} d^\beta \mu(k) \psi\left(\frac{y}{d^\beta}\right) \\ &= C_{1,\beta,1}(x, y) + C_{1,\beta,2}(x, y) + C_{1,\beta,3}(x, y), \end{aligned}$$

say, and where we have used the definition of $\psi(t)$. By using (2.36) and setting

$$\mathcal{E}(x, n) := \begin{cases} \log x & \text{if } n = 1, \\ 1, & \text{if } n > 1, \end{cases} \tag{2.85}$$

we can conclude that

$$C_{1,\beta,2}(x, y) = -\frac{1}{2} \sum_{dk \leqslant x} d^\beta \mu(k) = -\frac{x^{\beta+1}}{2(1+\beta)\zeta(1+\beta)} + O(x^\beta \mathcal{E}(x, \beta)).$$

The first sum is independent of β since we see that

$$C_{1,\beta,1}(x,y) = y\sum_{m\leqslant x}\sum_{k|m}\mu(k) = y.$$

Thus, it remains to compute $C_{1,3}^{(\beta)}(x,y)$ and this will require more effort. We begin by noting that

$$C_{1,\beta,3}(x,y) = \sum_{dk\leqslant x} d^\beta \mu(k)\psi\left(\frac{y}{d^\beta}\right) = \sum_{k\leqslant x}\mu(k)\sum_{d\leqslant x/k} d^\beta\psi\left(\frac{y}{d^\beta}\right).$$

Furthermore, we define the intervals $I_j := (N_j, 2N_j]$ where $N_j = N_{j,k} = \frac{x}{k}2^{-j}$ so that $1 \leqslant \frac{x}{k}2^{-j}$ implies that $j \ll \log x$. We may now write

$$\begin{aligned} C_{1,\beta,3}(x,y) &= \sum_{k\leqslant x}\mu(k)\sum_{j=1}^{\infty}\sum_{d\in I_j} d^\beta\psi\left(\frac{y}{d^\beta}\right) = \sum_{k\leqslant x}\mu(k)\sum_{j\ll\log x}\sum_{d\in I_j} d^\beta\psi\left(\frac{y}{d^\beta}\right) \\ &\leqslant \sum_{k\leqslant x}\sum_{j\ll\log x}\left|\sum_{d\in I_j} d^\beta\psi\left(\frac{y}{d^\beta}\right)\right|. \end{aligned}$$

The next step is to apply Abel summation to the inner sum to obtain

$$\begin{aligned} \sum_{d\in I_j} d^\beta\psi\left(\frac{y}{d^\beta}\right) &= \sum_{d\leqslant 2N_j} d^\beta\psi\left(\frac{y}{d^\beta}\right) \\ &= 2^\beta N_j^\beta\sum_{d\leqslant 2N_j}\psi\left(\frac{y}{d^\beta}\right) - \int_1^{2N_j}\sum_{d\leqslant t}\psi\left(\frac{y}{d^\beta}\right)\beta t^{\beta-1}dt \\ &\ll N_j^\beta\left|\sum_{d\leqslant 2N_j}\psi\left(\frac{y}{d^\beta}\right)\right| + \sup_{1\leqslant t\leqslant 2N_j}\left|\sum_{d\leqslant t}\psi\left(\frac{y}{d^\beta}\right)\right|(N_j^\beta - 1) \\ &\ll N_j^\beta\left|\sum_{d\leqslant 2N_j}\psi\left(\frac{y}{d^\beta}\right)\right|, \end{aligned}$$

Therefore we are left with

$$\sum_{d\in I_j} d^\beta\psi\left(\frac{y}{d^\beta}\right) \ll N_j^\beta \sup_{\mathbf{I}}\left|\sum_{d\in\mathbf{I}}\psi\left(\frac{y}{d^\beta}\right)\right|,$$

where the supremum is over all subintervals $\mathbf{I} = \{I_j,\ j = 1,\cdots,+\infty\}$. Thus, we have

$$C_{1,\beta,3}(x,y) \ll \sum_{k\leqslant x}\sum_{j=1}^{\infty} N_j^\beta \sup_{\mathbf{I}}\left|\sum_{n\in\mathbf{I}}\psi\left(\frac{y}{n^\beta}\right)\right|,$$

where we recall that the sum over j is finite and has $O(\log x)$ terms. Now we use Lemma 2.22. By taking $k = l = \frac{1}{2}$ and seeing that $f(n) = y/n^\beta \in \mathbf{F}(N, \infty, \beta+1, y, \varepsilon)$ the exponent pair estimate we need is

$$\sum_{n\in\mathbf{I}}\psi\left(\frac{y}{n^\beta}\right) \ll y^{\frac{1}{3}}N_j^{\frac{1-\beta}{3}} + y^{-1}N_j^{1+\beta}.$$

Consequently we have

$$\begin{aligned}
C_{1,\beta,3}(x,y) &\ll \sum_{k\leqslant x}\sum_{j=0}^{\infty} N_j^{\beta}(y^{\frac{1}{3}}N_j^{\frac{1-\beta}{3}} + y^{-1}N_j^{1+\beta}) \\
&= \sum_{k\leqslant x}\sum_{j=0}^{\infty}(y^{\frac{1}{3}}N_j^{\frac{1+2\beta}{3}} + y^{-1}N_j^{1+2\beta}) \\
&\ll \sum_{k\leqslant x}\left(y^{\frac{1}{3}}\frac{x^{\frac{1+2\beta}{3}}}{k^{\frac{1+2\beta}{3}}} + y^{-1}\frac{x^{1+2\beta}}{k^{1+2\beta}}\right) \\
&= x^{\frac{1+2\beta}{3}}y^{\frac{1}{3}}\mathcal{E}\left(x, \tfrac{1+2\beta}{3}\right) + x^{1+2\beta}y^{-1}.
\end{aligned}$$

This completes the proof.

2.9 Numerical evidence

The following plots illustrate the behavior of the first moment.

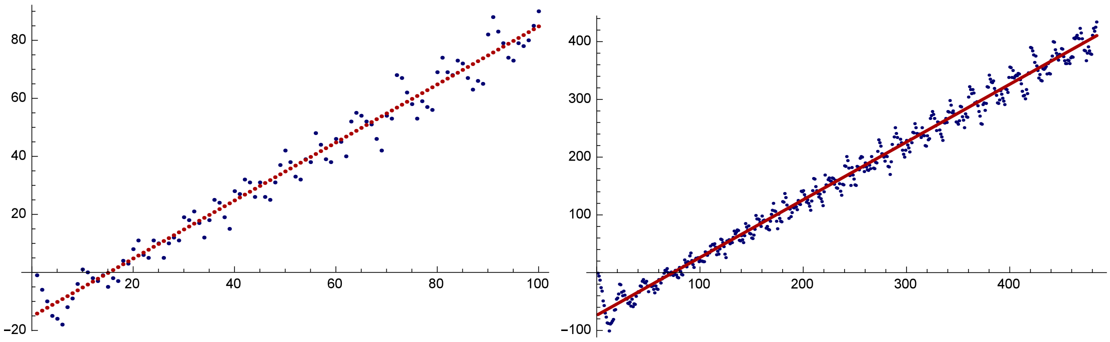

FIGURE 2.1: Left: in blue plof of $C_{1,\beta}(x,y)$ and in red main term right-hand side of (2.19) with $\beta = 1$, $x = 10$ and $y = x^{1+\beta}$. Right: same with $x = 22$.

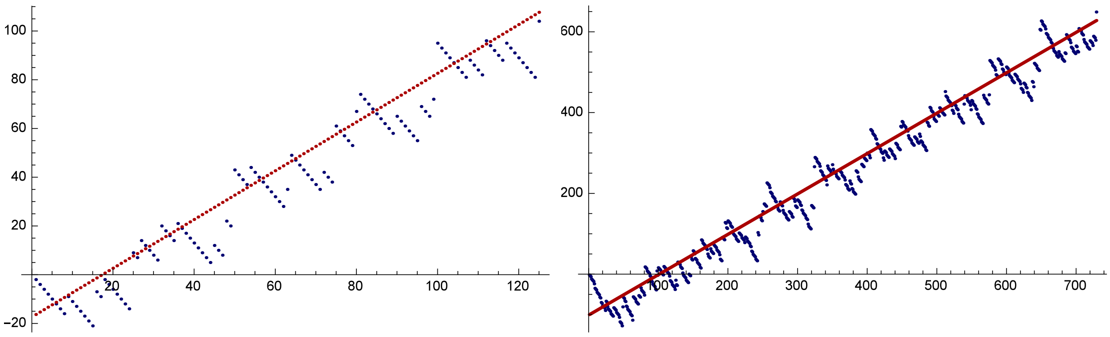

FIGURE 2.2: Left: in blue plof of $C_{1,\beta}(x,y)$ and in red main term right-hand side of (2.19) with $\beta = 2$, $x = 5$ and $y = x^{1+\beta}$. Right: same with $x = 9$.

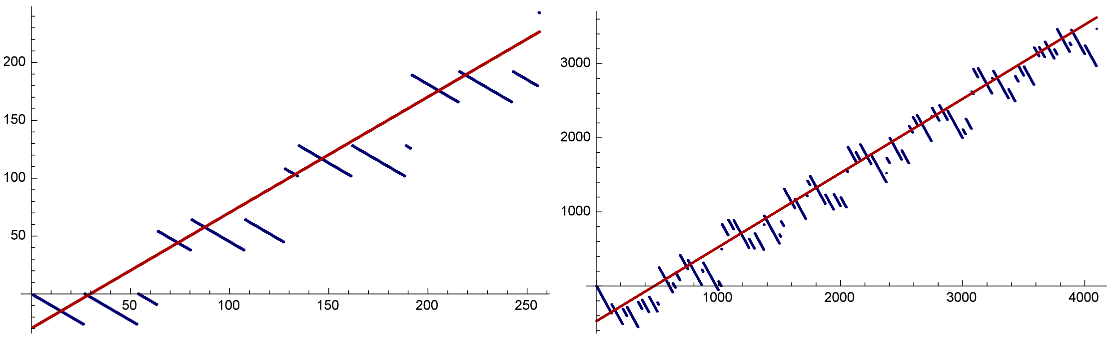

FIGURE 2.3: Left: in blue plof of $C_{1,\beta}(x,y)$ and in red main term right-hand side of (2.19) with $\beta = 3$, $x = 4$ and $y = x^{1+\beta}$. Right: same with $x = 8$.

Chapter 3

Zeros of combinations of the Riemann ξ-function on bounded vertical shifts

3.1 Introduction

The study of the zeros and the 'a-points' of the Riemann zeta-function is of special interest[1]. It is more difficult to locate the zeros or the 'a-points' than to study the value distributions of $\zeta(s)$.

The behavior of $\zeta(s)$ on every vertical line $\sigma = \mathrm{Re}(s) > \frac{1}{2}$ has been studied by Bohr and his collaborators. Let us take the half-plane $\sigma > \frac{1}{2}$, and remove all the points which have the same imaginary part as, and smaller real part than, one of the possible zeros (or the pole) of $\zeta(s)$ in this region. We denote the remaining part of this perforated half-plane by $\mathcal{G}$. Specifically, Bohr and Jessen [BJ30, BJ32] discovered that for $\sigma > \frac{1}{2}$, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mu\{\tau \in [0, T] : \sigma + i\tau \in \mathcal{G}, \log \zeta(\sigma + i\tau) \in \mathcal{R}\}$$

exists. Here μ is the Lebesgue measure and $\mathcal{R}$ is any fixed rectangle whose sides are parallel to the axes. Later Voronin [Vor72] provided a generalization of Bohr's denseness result.

> *For any fixed and distinct numbers $s_1, s_2, \ldots, s_n$ with $\frac{1}{2} < \mathrm{Re}(s_k) < 1$, the set $\{(\zeta(s_1 + it), \ldots, \zeta(s_n + it)) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$. Moreover, for any s with $\frac{1}{2} < \mathrm{Re}(s) < 1$, the set $\{(\zeta(s + it), \ldots, \zeta^{(n)}(s + it)) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$.*

Even more striking is Voronin's [Vor75] universality theorem.

[1] The solutions to $\zeta(s) = a$, which are denoted by $\rho_a = \beta_a + i\gamma_a$ are called a-points of $\zeta(s)$.

Let $0 < r < \frac{1}{4}$ and $g(s)$ be a non-zero analytic function on $|s| \le r$. Then for any $\epsilon > 0$, there exists a positive real number τ such that

$$\max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-g(s)\right| < \epsilon.$$

Moreover,

$$\liminf_{T\to\infty}\frac{1}{T}\mu\left\{\tau\in[0,T]:\max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-g(s)\right|<\epsilon\right\}>0.$$

Concerning Voronin's theorem, Bagchi [Bag81] gave an equivalent condition for the Riemann hypothesis of $\zeta(s)$. He proved in his doctoral thesis that

The Riemann hypothesis is true if, and only if, for any $\epsilon > 0$

$$\liminf_{T\to\infty}\frac{1}{T}\mu\left\{\tau\in[0,T]:\max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-\zeta(s)\right|<\epsilon\right\}>0.$$

These connections motivate us to study the vertical shifts $s \to s + i\tau$ of $\zeta(s)$. The values of $\zeta(s)$ on certain vertical arithmetic progressions have been studied by various authors. Putnam [Put54a, Put54b] showed that for any $d > 0$, the sequence $d, 2d, 3d, \cdots$ contains an infinity of elements which are not the imaginary parts of zeros of $\zeta(s)$ on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. More recently, van Frankenhuijesen [vF05] obtained bounds for the length of any hypothetical arithmetic progression. Good [Goo78] found asymptotic formulae for the discrete second and fourth moments of $\zeta(s)$ on arbitrary arithmetic progressions to the right of the critical line. Steuding and Wegert [SW12] succeeded in obtaining an asymptotic formula for the discrete first moment of $\zeta(s)$ on arbitrary arithmetic progressions in the critical strip. Li and Radziwill [LRar] established, among many other things, results on the distribution of values of $\zeta(\frac{1}{2} + i(al + b))$ as l ranges over the integers in some dyadic interval $[T, 2T]$.

Define functions η and ρ in terms of the Riemann zeta-function as

$$\eta(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{and} \quad \rho(t) := \eta\left(\frac{1}{2}+it\right).$$

It is well-known that $\eta(s)$ is a meromorphic function with poles at $s = 0$ and 1, and that $\rho(t)$ is a real-valued function. Hardy [Har14] proved that $\rho(t)$ has infinitely many real zeros. In other words, the Riemann zeta-function has infinitely many zeros on the critical line.

Let $\{c_j\}$ be a sequence of real numbers such that

$$\sum_{j=1}^{\infty}|c_j| < \infty.$$

Let $\{\lambda_j\}$ be a bounded sequence of real numbers. Define

$$F(s) := \sum_{j=1}^{\infty} c_j\eta(s+i\lambda_j). \tag{3.1}$$

Note that $F(s)$ has poles at $-i\lambda_j$ and $1-i\lambda_j$ for all j. Using Stirling's formula for the gamma function (A.6) and the boundedness of the sequence $\{\lambda_j\}$, one can show that there exists a bounded set $D \subset \mathbb{C}$ such that $F(s)$ is analytic on $\mathbb{C} \setminus D$. In particular, D can be taken as the union of two bounded vertical intervals containing the points 0 and 1. If $\{\lambda_j\}$ is an infinite sequence, then $F(s)$ has essential singularities inside the set D. Independently of whether $F(s)$ has essential singularities or not, it can be seen that $F(\frac{1}{2}+it)$ is a well-defined, real-valued function for $t \in \mathbb{R}$. A natural question arises - what can we say about the zeros of $F(s)$?

Without loss of generality, we can take $c_j \neq 0$ and the λ_j's to be distinct for all j. Indeed, the terms corresponding to c_j's being zero have no contribution to $F(s)$. Furthermore, since the right-hand side of (3.1) is absolutely convergent, the terms for which $\lambda_i = \lambda_j$ can be grouped together and we can denote the new coefficient by c_j. We have the following result.

Theorem 3.1. *Let $\{c_j\}$ be a sequence of non-zero real numbers so that $\sum_{j=1}^{\infty} |c_j| < \infty$. Let $\{\lambda_j\}$ be a bounded sequence of distinct real numbers that attains its bounds. Then the function $F(s) = \sum_{j=1}^{\infty} c_j \eta(s+i\lambda_j)$ has infinitely many zeros on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$.

3.2 Preliminaries

The following lemmas are instrumental in the proofs of our theorems.

Lemma 3.2. *For $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$,*

$$\int_{-\infty}^{\infty} e^{\alpha t}\rho(t)dt = -4\pi\cos\frac{\alpha}{2} + 2\pi e^{\frac{\alpha i}{2}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}. \tag{3.2}$$

For details see Landau [Lan15, section 3] or Remark 4.1 below.

Lemma 3.3. *Let $h : \mathbb{C} \to \mathbb{C}$ be analytic at $\alpha = \frac{\pi}{4}$. As $\alpha \to \frac{\pi}{4}$, we have*

$$\frac{d^m}{d\alpha^m}\left(h(\alpha)\sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}\right) \to 0. \tag{3.3}$$

The above result can be adapted from Landau [Lan15, section 3] and Titchmarsh [Tit86, p. 257].

Proof. Let $\hat{\vartheta}_3(\delta)$ be defined as in (4.2). Then $\hat{\vartheta}_3$ is analytic for $-\frac{\pi}{2} < \arg\delta < \frac{\pi}{2}$. Note that

$$\hat{\vartheta}_3(i+\delta) = \sum_{n=1}^{\infty} e^{-n^2\pi(i+\delta)} = \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi\delta}.$$

Next we have

$$\hat{\vartheta}_3(4\delta) = \sum_{n=1}^{\infty} e^{-4\delta n^2\pi} = \sum_{n=1}^{\infty} e^{-(2n)^2\pi\delta} = \sum_{m\in 2\mathbb{N}} e^{-m^2\pi\delta},$$

as well as

$$\hat{\vartheta}_3(\delta) = \sum_{m\in 2\mathbb{N}} e^{-m^2\pi\delta} + \sum_{m\in 2\mathbb{N}+1} e^{-m^2\pi\delta}.$$

Therefore,

$$2\hat{\vartheta}_3(4\delta) - \hat{\vartheta}_3(\delta) = \sum_{m\in 2\mathbb{N}} e^{-m^2\pi\delta} - \sum_{m\in 2\mathbb{N}+1} e^{-m^2\pi\delta} = \sum_{m\in \mathbb{N}} (-1)^m e^{-m^2\pi\delta} = \hat{\vartheta}_3(i+\delta).$$

Using (4.3), we have

$$\begin{aligned}\hat{\vartheta}_3(i+\delta) &= \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{4\delta}\right) + \frac{1}{2\sqrt{\delta}} - 1 - \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{\delta}\right) - \frac{1}{2}\frac{1}{\sqrt{\delta}} + \frac{1}{2}\\ &= \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}}\hat{\vartheta}_3\left(\frac{1}{\delta}\right) - \frac{1}{2}.\end{aligned}$$

Since $\exp(-1/x)$ tends to zero as $x \to 0$ faster than any power x^{-k} going to infinity (for $k > 0$), we see that as $\delta \to 0^+$, the function $\frac{1}{2} + \hat{\vartheta}_3(i+\delta)$ and all of its derivatives tend to zero. Since $h(\alpha)$ is analytic at $\pi/4$ and

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}} = 1 + 2\hat{\vartheta}_3(e^{2\alpha i}),$$

if $\alpha \to \frac{1}{4}\pi^+$, i.e., $e^{2i\alpha} \to i$ along any path in the wedge $|\arg(e^{2i\alpha} - i)| < \frac{1}{2}\pi$,

$$\lim_{\alpha\to\frac{1}{4}\pi^+} \frac{d^m}{d\alpha^m}[h(\alpha)(1 + 2\hat{\vartheta}_3(e^{2i\alpha}))] = 0.$$

This proves the lemma. $\square$

The following result is due to Kronecker (see Hardy and Wright [HW54]).

Lemma 3.4. *Let $(n\theta)$ denote the fractional part of $n\theta$. If θ is irrational, then the set of points $(n\theta)$ is dense in the interval $(0, 1)$.*

Remark 3.5. If θ is rational, then the set of points $(n\theta)$ is periodic in the interval $(0, 1)$.

3.3 Zeros of $F(s)$: Proof of Theorem 3.1

Let λ_j be any real number. Replacing t by $t + \lambda_j$ in Lemma 3.2 we find

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\alpha t}\rho(t+\lambda_j)dt &= e^{-\alpha\lambda_j}\left(-4\pi\cos\frac{\alpha}{2} + 2\pi e^{\frac{\alpha i}{2}}\sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}\right) \qquad (3.4)\\ &= -2\pi\left(e^{\frac{\alpha i}{2}-\alpha\lambda_j} + e^{-\frac{\alpha i}{2}-\alpha\lambda_j} - e^{\frac{\alpha i}{2}-\alpha\lambda_j}\sum_{n=-\infty}^{\infty} e^{-n^2\pi e^{2\alpha i}}\right).\end{aligned}$$

Differentiating both sides of (3.4) $2m$ times with respect to α we get

$$\int_{-\infty}^{\infty} t^{2m}e^{\alpha t}\rho(t+\lambda_j)dt = -2\pi\left(\left(\frac{i}{2}-\lambda_j\right)^{2m} e^{\frac{\alpha i}{2}-\alpha\lambda_j} + \left(\frac{i}{2}+\lambda_j\right)^{2m} e^{-\frac{\alpha i}{2}-\alpha\lambda_j}\right. \qquad (3.5)$$

$$- \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) \Bigg).$$

Let $\frac{i}{2} - \lambda_j = r_j e^{i\theta_j}$. Without loss of generality, one may take $0 < \theta_j < \frac{\pi}{2}$. From (3.5) we have

$$\begin{aligned} \int_{-\infty}^{\infty} t^{2m} e^{\alpha t} \rho(t+\lambda_j) dt &= -2\pi e^{-\alpha\lambda_j} \left(r_j^{2m} e^{i(\frac{\alpha}{2} + 2m\theta_j)} + r_j^{2m} e^{i(\frac{-\alpha}{2} + 2\pi m - 2m\theta_j)} \right) \\ &\quad + 2\pi \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) \\ &= -4\pi e^{-\alpha\lambda_j} r_j^{2m} \cos\left(\frac{\alpha}{2} + 2m\theta_j \right) \\ &\quad + 2\pi \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right). \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by c_j and summing over j, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} t^{2m} e^{\alpha t} F\left(\frac{1}{2} + it \right) dt &= -4\pi \sum_{j=1}^{\infty} c_j e^{-\alpha\lambda_j} r_j^{2m} \cos\left(\frac{\alpha}{2} + 2m\theta_j \right) \\ &\quad + 2\pi \sum_{j=1}^{\infty} \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(c_j e^{\frac{\alpha i}{2} - \alpha\lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) \\ &=: -4\pi \sum_{j=1}^{\infty} h_j(\alpha) + 2\pi \sum_{j=1}^{\infty} \tilde{h}_j(\alpha). \end{aligned} \tag{3.7}$$

By Stirling's formula (A.6) we have

$$\rho(t) \ll |t|^A e^{-\frac{\pi}{4}|t|}$$

as $t \to \infty$, where A is a positive number. Since $\{\lambda_j\}$ is a bounded sequence, we find that

$$\sum_{j=1}^{\infty} c_j \rho(t+\lambda_j) \ll |t|^A e^{-\frac{\pi}{4}|t|} \sum_{j=1}^{\infty} |c_j| \ll |t|^A e^{-\frac{\pi}{4}|t|} \tag{3.8}$$

as $t \to \infty$. Hence (3.8) justifies the interchange of summation and integration on the left-hand side of (3.7) for $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$. Also for the same bounded sequence $\{\lambda_j\}$, and for any given bounded interval I_0 with $\alpha \in I_0$, we have

$$||h_j(\alpha)||_\infty \le \sum_{j=1}^{\infty} |c_j| \max_{\alpha, j} \{ r_j^{2m} e^{-\alpha\lambda_j} \} \ll 1 \tag{3.9}$$

uniformly for $\alpha \in I_0$.

Let m be a fixed non-negative integer. Let $\epsilon > 0$ be any number such that $\frac{\pi}{4} - \epsilon \le \alpha \le \frac{\pi}{4} + \epsilon$. For any $0 \le l \le m$, we observe that

$$\sum_{n=-\infty}^{\infty} n^{2l} e^{-n^2 \pi \cos(2\alpha)} \ll_\epsilon 1, \tag{3.10}$$

and for any bounded sequence $\{\lambda_j\}$,

$$\frac{\partial^l}{\partial \alpha^l} e^{\frac{\alpha i}{2} - \alpha \lambda_j} \ll_{\epsilon,m} 1. \tag{3.11}$$

Using Leibniz's rule, (3.10) and (3.11), we have $|\tilde{h}_j(\alpha)| \ll_{\epsilon,m} |c_j|$. Hence

$$||\tilde{h}_j(\alpha)||_\infty \ll_{\epsilon,m} 1 \tag{3.12}$$

for $\frac{\pi}{4} - \epsilon \leq \alpha \leq \frac{\pi}{4} + \epsilon$. Let

$$h(\alpha) := \sum_{j=1}^{\infty} c_j e^{-\alpha \lambda_j}.$$

Note that $h(\alpha)$ is an entire function. From (3.12), we find that

$$\sum_{j=1}^{\infty} \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(c_j e^{\frac{\alpha i}{2} - \alpha \lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) = \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(h(\alpha) e^{\frac{\alpha i}{2}} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right)$$

for $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$. Therefore by Lemma 3.3, we deduce that

$$\lim_{\alpha \to \frac{\pi}{4}^+} \sum_{j=1}^{\infty} \frac{\partial^{2m}}{\partial \alpha^{2m}} \left(c_j e^{\frac{\alpha i}{2} - \alpha \lambda_j} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi e^{2\alpha i}} \right) = 0. \tag{3.13}$$

Letting $\alpha \to \frac{\pi}{4}^+$ on both sides of (3.7) and using (3.9) and (3.13) we get

$$\lim_{\alpha \to \frac{\pi}{4}^+} \int_{-\infty}^{\infty} t^{2m} e^{\alpha t} F\left(\frac{1}{2} + i(t + \lambda_j)\right) dt = -4\pi \sum_{j=1}^{\infty} c_j e^{\frac{-\pi \lambda_j}{4}} r_j^{2m} \cos\left(\frac{\pi}{8} + 2m\theta_j\right). \tag{3.14}$$

By hypothesis, there exists a positive integer M such that

$$|\lambda_M| = \max_j \{|\lambda_j|\} \quad \text{and} \quad \lambda_M \neq \lambda_j \quad \text{for} \quad M \neq j.$$

Then the right-hand side of (3.14) can be written as

$$-4\pi c_M r_M^{2m} e^{-\frac{\pi \lambda_M}{4}} \cos\left(\frac{\pi}{8} + m\theta_M\right)(1 + E(X) + H(X)), \tag{3.15}$$

where

$$E(X) := \sum_{\substack{j \neq M \\ j \leq X}} \frac{c_j}{c_M} e^{-\frac{\pi}{4}(\lambda_j - \lambda_M)} \left(\frac{r_j}{r_M}\right)^{2m} \frac{\cos(\frac{\pi}{8} + 2m\theta_j)}{\cos(\frac{\pi}{8} + 2m\theta_M)}, \tag{3.16}$$

as well as

$$H(X) := \sum_{\substack{j \neq M \\ j > X}} \frac{c_j}{c_M} e^{-\frac{\pi}{4}(\lambda_j - \lambda_M)} \left(\frac{r_j}{r_M}\right)^{2m} \frac{\cos(\frac{\pi}{8} + 2m\theta_j)}{\cos(\frac{\pi}{8} + 2m\theta_M)}. \tag{3.17}$$

Next we claim that there exists a sequence such that for each value m in it, the inequality $|\cos(\frac{\pi}{8}+2m\theta_M)|\geq \frac{1}{3}$ holds. Let $\frac{i}{2}-\lambda_M = r_M e^{i\theta_M}$ for $0<\theta_M<\frac{\pi}{2}$. Then

$$r_M > r_j \quad \text{for} \quad M \neq j. \tag{3.18}$$

Now we divide the proof of the claim into two cases. First consider the case when $\frac{\theta_M}{\pi}$ is irrational. Then by Lemma 3.4, we find two subsequences $\{p_n\}$ and $\{q_n\}$ such that

$$\left(\frac{p_n\theta_M}{\pi}\right)\to\frac{1}{2^4} \quad \text{and} \quad \left(\frac{q_n\theta_M}{\pi}\right)\to\frac{1}{2}$$

for $n\to\infty$, where, as before, (x) denotes the fractional part of x. One can see that for $n\to\infty$,

$$\cos\left(\frac{\pi}{8}+p_n\theta_M\right)\to\frac{1}{\sqrt{2}} \quad \text{and} \quad \cos\left(\frac{\pi}{8}+q_n\theta_M\right)\to -\cos\left(\frac{\pi}{8}\right)<-\frac{1}{3}. \tag{3.19}$$

In the second case, we consider $\frac{\theta_M}{\pi}:=\frac{p}{q}$ to be rational. Since $0<\frac{p}{q}<\frac{1}{2}$, there exists an integer n_0 such that $\frac{1}{4}\leq n_0\lfloor\frac{q}{2p}\rfloor\frac{p}{q}<\frac{1}{2}$. Now define $p_n:=nq$ and $q_n:=nq+n_0\lfloor\frac{q}{2p}\rfloor$. Therefore for all n,

$$\cos\left(\frac{\pi}{8}+p_n\theta_M\right)=\cos\left(\frac{\pi}{8}\right) \quad \text{and} \quad -\cos\left(\frac{\pi}{8}\right)<\cos\left(\frac{\pi}{8}+q_n\theta_M\right)\leq\cos\left(\frac{5\pi}{8}\right). \tag{3.20}$$

The above constructions show that if m runs through the sequence $\{p_n\}\cup\{q_n\}$, then for large m, we have

$$\left|\cos\left(\frac{\pi}{8}+2m\theta_M\right)\right|\geq\frac{1}{3}. \tag{3.21}$$

Let m be any large integer from the sequence $\{p_n\}\cup\{q_n\}$. From (3.17), (3.18) and (3.21) we have

$$H(X)\leq\frac{3}{|c_M|}\sum_{\substack{j\neq M\\ j>X}}|c_j|<\frac{1}{1914}, \tag{3.22}$$

for a large X. Let

$$c_X=\max_{j\leq X}\left\{\frac{|r_j|}{|r_M|}\right\}.$$

Since X is finite, by (3.18) we find that $c_X<1$. Similarly for large $m\in\{p_n\}\cup\{q_n\}$, using (3.16) and (3.21) we have

$$E(X)\leq 3\frac{c_X^{2m}}{|c_M|}\sum_{\substack{j\neq M\\ j\leq X}}|c_j|, \tag{3.23}$$

which tends to 0 as $m\to\infty$ through the sequence $\{p_n\}\cup\{q_n\}$. Also by construction $\cos(\frac{\pi}{8}+m\theta_M)$ changes sign infinitely often for infinitely many values of $m\in\{p_n\}\cup\{q_n\}$. Hence from (3.15), (3.22), and (3.23), we see that the right-hand side of (3.14) changes sign infinitely often for infinitely many values of $m\in\{p_n\}\cup\{q_n\}$.

Let us now suppose that $F(s)$ has only finitely many zeros on the line $\sigma = \frac{1}{2}$, and hence that $F(\frac{1}{2}+it)$ never changes sign for $|t| > T$ for some large T. In other words, we can say that $F(\frac{1}{2}+it) > 0$ for $|t| > T$, or that $F(\frac{1}{2}+it) < 0$ for $|t| > T$, or that $F(\frac{1}{2}+it)$ takes opposite signs in $t > T$ and $t < T$. First of all, let us consider that $F(\frac{1}{2}+it) > 0$ for $|t| > T$.

Next, let us define the quantity L by the equation

$$L := \lim_{\alpha\to\frac{\pi}{4}^+} \int_{|t|\geq T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\alpha t} dt. \tag{3.24}$$

Since the integrand in the above integral is positive, one sees that for any $T' > T$,

$$\lim_{\alpha\to\frac{\pi}{4}} \int_{T\leq|t|\leq T'} F\left(\frac{1}{2}+it\right) t^{2m} e^{\alpha t} dt \leq \lim_{\alpha\to\frac{\pi}{4}} \int_{T}^{\infty} F\left(\frac{1}{2}+it\right) t^{2m} e^{\alpha t} dt = L. \tag{3.25}$$

In particular,

$$\int_{T\leq|t|\leq T'} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t} dt \leq L. \tag{3.26}$$

Hence

$$\int_{-\infty}^{\infty} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t} dt \tag{3.27}$$

is convergent.

Thus, for every $m \in \mathbb{N}$ we have

$$\int_{-\infty}^{\infty} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t} dt = -4\pi \sum_{j=1}^{\infty} c_j e^{\frac{-\pi\lambda_j}{4}} r_j^{2m} \cos\left(\frac{\pi}{8} + 2m\theta_j\right). \tag{3.28}$$

This is impossible since the right-hand side switches sign infinitely often. We can find an integer $m \in \{p_n\} \cup \{q_n\}$ such that

$$\begin{aligned} \int_{|t|\geq T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t} dt &< -\int_{-T}^{T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t} dt \\ &< T^{2m} \int_{-T}^{T} \left| F\left(\frac{1}{2}+it\right) e^{\frac{\pi}{4}t} \right| dt \\ &\leq T^{2m} R. \end{aligned} \tag{3.29}$$

It is seen that R is independent of m.

Finally, by the hypothesis on $F(\frac{1}{2}+it)$, we see that there exists $\varepsilon = \varepsilon(T) > 0$ such that $F(\frac{1}{2}+it) \geq \varepsilon$ for all $2T < t < 2T+1$. Hence

$$\begin{aligned} \int_{|t|\geq T} F\left(\frac{1}{2}+it\right) t^{2m} e^{\frac{\pi}{4}t} dt &\geqslant \int_{2T}^{2T+1} \varepsilon t^{2m} e^{\frac{\pi}{4}t} dt \\ &\geqslant \int_{2T}^{2T+1} \varepsilon t^{2m} dt \end{aligned} \tag{3.30}$$

$$= \varepsilon\left(\frac{(2T+1)^{2m+1}}{2m+1} - \frac{(2T)^{2m+1}}{2m+1}\right)$$
$$\geqslant \varepsilon(2T)^{2m}.$$

Combining these two results, we have

$$\varepsilon(2T)^{2m} \leqslant \int_{|t|\geq T} F\left(\frac{1}{2}+it\right)t^{2m}e^{\frac{\pi}{4}t}dt < -\int_{-T}^{T} F\left(\frac{1}{2}+it\right)t^{2m}e^{\frac{\pi}{4}t}dt < T^{2m}R, \quad (3.31)$$

for infinitely values of $m \in \{p_n\} \cup \{q_n\}$. This is equivalent to

$$2^{2m} < \frac{R}{\varepsilon}$$

holding for infinitely many values of $m \in \{p_n\} \cup \{q_n\}$. However this is impossible since m can be taken to be arbitrarily large.

Now, if $F(\frac{1}{2}+it) < 0$ for $|t| > T$, we multiply both sides of (3.14) by -1 and carry out similar arguments for $-F(\frac{1}{2}+it)$. Lastly, if $F(\frac{1}{2}+it)$ takes opposite signs in $t > T$ and $t < -T$ then we differentiate (3.4) $2m+1$ times with respect to α, instead of $2m$ times. In this case, (3.14) and (3.18) can be proved similarly. If $\frac{\theta_M}{\pi}$ is irrational, then we construct the sequences $\{p_n\}$ and $\{q_n\}$ such that

$$\left(\frac{p_n\theta_M}{\pi}\right) \to \frac{1}{2^4} - \frac{\theta_M}{2\pi} \quad \text{and} \quad \left(\frac{q_n\theta_M}{\pi}\right) \to \frac{1}{2} - \frac{\theta_M}{2\pi}.$$

If $\frac{\theta_M}{\pi}$ is rational, then we construct the sequences $\{p_n\}$ and $\{q_n\}$ by

$$p_n := nq - \frac{1}{2} \quad \text{and} \quad q_n := nq + n_0\left\lfloor \frac{q}{2p} \right\rfloor - \frac{1}{2}.$$

For the above $\{p_n\} \cup \{q_n\}$ one can check easily that (3.19), (3.20) and (3.21) hold when we replace $2m$ by $2m+1$. Likewise, one can use similar arguments to prove (3.31) for $2m+1$ and arrive at a contradiction. Hence we have proved the theorem.

Chapter 4

Integrals involving the Riemann-Ξ function

4.1 Introduction

Let functions ξ and Ξ be defined in terms of ζ by

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{and} \quad \Xi(t) := \xi\left(\frac{1}{2}+it\right).$$

One of the essential ingredients in the proof of Theorem 3.1 was the integral identity

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cos(\nu t)dt = \frac{\pi}{2}(e^{\frac{\nu}{2}} - 2e^{-\frac{\nu}{2}}\hat{\vartheta}_3(e^{-2\nu})), \tag{4.1}$$

which was used by Hardy as well to prove that the Riemann zeta-function has infinitely many zeros on the critical line (see [Har14], and §2.16 of [Tit86]). Here

$$\hat{\vartheta}_3(x) := \sum_{n=1}^{\infty}\exp(-x\pi n^2). \tag{4.2}$$

Note that for $\mathrm{Re}(x) > 0$, we have the functional equation

$$2\hat{\vartheta}_3(x) + 1 = x^{-\frac{1}{2}}(2\hat{\vartheta}_3(x^{-1}) + 1). \tag{4.3}$$

Identities such as (4.1) have many applications. For example, using Fourier's integral theorem, we can invert identities like (4.1) to obtain new expressions for the Riemann Ξ-function, and hence for the Riemann zeta-function. For instance, if we replace ν by $-\nu$ in (4.1) and use Fourier's integral theorem, we have

$$\Xi(t) = \left(t^2+\frac{1}{4}\right)\int_0^\infty (e^{-\frac{u}{2}} - e^{\frac{u}{2}}\hat{\vartheta}_3(e^{2u}))\cos(tu)du,$$

which is well-known [Tit86, p. 254, Equation (10.1.1)]. More on this integral representation can be found in [Rie59] and [Edw74, §1.8]. New integral identities of the type (4.1), having the function $\Xi(t)$ under the integral sign, were studied by Ramanujan [Ram15, Equations (12), (16)] (see also[Har17, p. 37]) and later by Koshliakov [Kos37, Equations

(18), (25) and (38)], [Kos34, p. 404–405]. For recent work in this direction, see the survey article due to Dixit [Dix13c]. At this juncture, it is important to note the following quote by Hardy [Har17] regarding Ramanujan's work on these integrals:

> *... The unsolved problems concerning the zeros of* $\zeta(s)$ *or of* $\Xi(t)$ *are among the most obscure and difficult in the whole range of Pure Mathematics. Any new formulae involving* $\zeta(s)$ *and* $\Xi(t)$ *are of very great interest, because of the possibility that they may throw new light on some outstanding questions...*

Another goal of this thesis is to generalize identity (4.1) and other allied identities by replacing the term $\cos(xt)$ in (4.1) by a more general class of functions that will be discussed shortly. These allied identities are similar in nature to (4.1) in that they involve integrals of the form

$$\int_0^\infty f\left(\frac{t}{2}\right)\Xi^k\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\alpha\right)dt,$$

where $f(t)$ is of the form $f(t)=g(it)g(-it)$ with g analytic in t and $k=1$ or $k=2$. Four well-known examples of such identities are those due to Ferrar [Fer36] and Hardy [Har15, Equation (2)] both of which have $k=1$, and due to Koshliakov [Kos29, Fer36] and Ramanujan [Ram15, Ram88] for $k=2$.

Corollary 4.1 (Hardy integral theorem)**.** *For* $\theta>0$ *one has*

$$2\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2}\frac{\cos(\frac{1}{2}t\log\theta)}{\cosh\frac{1}{2}\pi t}dt=\sqrt{\theta}\int_0^\infty e^{-\pi\theta^2x^2}(\psi(x+1)-\log x)dx, \tag{4.4}$$

where $\psi(x)$ *denotes the logarithmic derivative of the gamma function* $\Gamma(x)$*.*

Corollary 4.2 (Ferrar integral theorem)**.** *For* $\theta>0$ *one has*

$$\begin{aligned}\frac{-1}{\sqrt{\pi}}\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(\frac{t}{2})}{1+t^2}\cos\left(\frac{1}{2}t\log\theta\right)dt&\\ =\sqrt{\theta}\int_0^\infty e^{-\frac{\theta^2t^2}{4\pi}}\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)dt,&\end{aligned} \tag{4.5}$$

where $K_\nu(z)$ *denotes the modified Bessel function of order* ν*.*

Corollary 4.3 (Koshliakov's integral theorem)**.** *For* $\theta>0$ *one has*

$$-\frac{32}{\pi}\int_0^\infty \Xi^2\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\theta\right)\frac{dt}{(1+t^2)^2}=\sqrt{\theta}\left(\frac{\gamma-\log(4\pi\theta)}{\theta}-4\sum_{n=1}^\infty d(n)K_0(2\pi n\theta)\right), \tag{4.6}$$

Corollary 4.4 (Ramanujan integral theorem)**.** *For* $\theta>0$ *one has*

$$\begin{aligned}-\frac{1}{\pi^{3/2}}\int_0^\infty\left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2\Xi^2\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\theta\right)\frac{dt}{1+t^2}&\\ =\sqrt{\theta}\left\{\frac{\gamma-\log 2\pi\theta}{2\theta}+\sum_{n=1}^\infty\lambda(n\theta)\right\},&\end{aligned} \tag{4.7}$$

where $\lambda(x)$ *is given by*

$$\lambda(x) := \frac{\Gamma'}{\Gamma}(x) + \frac{1}{2x} - \log x.$$

These four identities are now derived as special cases of more general results given below, namely Theorems 4.10, 4.11, 4.19 and 4.20, respectively.

Moreover, there is an additional way of thinking about these integrals. As explained in Chapter 1, explicit formulae relate the sum of a suitable function over prime powers to the sum of the Mellin transform of that function over the zeros of the Riemann zeta-function, see [Tit86, §14.20].

In this chapter, we deal with integrals, rather than sums; however in the integrands we have the function $\Xi(t)$, whose sign changes encapsulate information on the non-trivial zeros of the Riemann zeta-function. On the other hand, the arithmetical information is to be found in the divisor function $d(n)$. Therefore, although Corollaries 4.1 to 4.4 are not explicit formulae, one may argue that the integral on the left-hand side of each of these results captures some of the behavior of the non-trivial zeros in a "continuous" analogue of the summatory explicit formulae, which are of discrete type.

The reciprocity of two functions and their use in the theory of the Riemann zeta-function was indeed first observed by Ramanujan as credited by Hardy and Littlewood in [HL18]. Specifically, Ramanujan first noted that under certain conditions, which are explained in Chapter 1, one has the explicit formula (1.4). Ramanujan then noted that if φ and ψ are Fourier cosine reciprocal functions, then one had the substantially more general (1.3). The particular case (1.4) then follows by taking the Fourier cosine reciprocal pair

$$\varphi(x) = \psi(x) = \exp(-x^2).$$

This is the same mechanism that is used in this chapter (albeit more complex when it comes to $\Xi^2(t)$) to generalize the above integral identities. It is noteworthy to remark that Ramanujan was able to see the reciprocity of φ and ψ in the sums appearing in explicit formulas such as (1.3), but for some reason he did not publish[1] (though he might certainly have foreseen it) how ubiquitous these reciprocities can be in integrals such as (4.7) or (4.4) of which he must have been well-aware[2].

4.2 One power of Ξ: Hardy and Ferrar

Suppose the functions ϕ and ψ are reciprocal in the Fourier cosine transform as in Chapter 1, i.e.

$$\frac{\sqrt{\pi}}{2}\phi(x) = \int_0^\infty \psi(u)\cos(2ux)du \quad \text{and} \quad \frac{\sqrt{\pi}}{2}\psi(x) = \int_0^\infty \phi(u)\cos(2ux)du. \tag{4.8}$$

We now define $Z_1(s)$ and $Z_2(s)$ in terms of the Mellin transforms of ϕ and ψ by

$$\Gamma\left(\frac{s}{2}\right)Z_1(s) := \int_0^\infty x^{s-1}\phi(x)dx \quad \text{and} \quad \Gamma\left(\frac{s}{2}\right)Z_2(s) := \int_0^\infty x^{s-1}\psi(x)dx, \tag{4.9}$$

[1] This statement is made to the best of the author's knowledge.

[2] See [Bha97] for Bhaskaran's remarks on the pervasiveness of reciprocity of functions in Ramanujan's work on Fourier integrals and Tauberian theory.

each valid in a specific vertical strip in the complex s-plane. Note that in case of a non-empty intersection of the two corresponding vertical strips, the Mellin inversion theorem of (A.27) gives

$$\phi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s}{2}\right) Z_1(s) x^{-s} ds \quad \text{and} \quad \psi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma\left(\frac{s}{2}\right) Z_2(s) x^{-s} ds, \tag{4.10}$$

where $\mathrm{Re}(s) = c$ lies in the intersection. Moreover, let us define

$$\Theta(x) := \phi(x) + \psi(x) \quad \text{and} \quad Z(s) := Z_1(s) + Z_2(s) \tag{4.11}$$

so that

$$\Gamma\left(\frac{s}{2}\right) Z(s) = \int_0^\infty x^{s-1}\Theta(x) dx \tag{4.12}$$

for values of s in the intersection of the two strips. Let us make an alteration on the class K of Chapter 1.

Definition 4.5. Let $0 < \omega \le \pi$ and $\lambda < \frac{1}{2}$. If $f(z)$ is such that

i) $f(z)$ is analytic with $z = re^{i\theta}$, regular in the angle defined by $r > 0$, $|\theta| < \omega$,

ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O(|z|^{-\lambda-\varepsilon}) & \text{if } |z| \text{ is small,} \\ O(|z|^{-\beta-\varepsilon}) & \text{if } |z| \text{ is large,} \end{cases} \tag{4.13}$$

for every $\varepsilon > 0$ and $\beta > \lambda$, and uniformly in any angle $\theta < \omega$,

then we say that f belongs to the class $\clubsuit$ and write $f(z) \in \clubsuit(\omega, \lambda, \beta)$.

Lemma 4.6. *Let $\phi, \psi \in \clubsuit(\omega, 0, \alpha)$ and Z be defined by (4.11). Then we have*

$$Z(1-s) = Z(s).$$

Proof. We know ϕ and ψ are cosine reciprocal which is a special case of functions reciprocal in the Hankel transform when $\nu = -1/2$. Then by a minor modification of Lemma 1.9 of Chapter 1 we have

$$Z_2(s) = Z_1(1-s) \quad \text{and} \quad Z_1(s) = Z_2(1-s),$$

where Z_1 and Z_2 are defined by (4.9). Finally, note that

$$Z(s) = Z_1(s) + Z_2(s) = Z_1(1-s) + Z_2(1-s) = Z(1-s), \tag{4.14}$$

as claimed. □

Our next results are as follows.

Theorem 4.7. *Let $\beta > 1$ and $\phi, \psi \in \clubsuit(\omega, 0, \beta)$ and suppose that Θ and Z are defined as in (4.11). Then we have*

$$\int_0^\infty \frac{\Xi(t)}{t^2 + \frac{1}{4}} Z\left(\frac{1}{2} + it\right) dt = \frac{\pi}{2} Z(1) - \frac{\pi}{2}\sum_{n=1}^\infty \Theta(\pi^{1/2} n). \tag{4.15}$$

Corollary 4.8. *Identity* (4.1) *is a special case of Theorem* 4.7 *with* $\phi(x) = \psi(x) = \exp(-e^{2\nu}x^2)$ *for* $-\frac{\pi}{4} < \mathrm{Im}(\nu) < \frac{\pi}{4}$.

The following result, which was obtained by Dixit in [Dix13b] to study a generalization of the identity (4.1), is also a special case of Theorem 4.7.

Corollary 4.9 (Dixit integral theorem)**.** *Let* $h \in \mathbb{C}$ *be fixed and* x *such that* $-\frac{\pi}{4} < x < \frac{\pi}{4}$. *If we set*

$$\varpi(h,x) := \sum_{n=1}^{\infty} \exp(-\pi x n^2)\cos(\pi^{1/2}hn), \tag{4.16}$$

and

$$\nabla(h,t,e^{-2x}) := e^{xit}{}_1F_1\left(\frac{1}{4}+\frac{1}{2}it;\frac{1}{2};-\left(\frac{he^x}{2}\right)^2\right) + e^{-xit}{}_1F_1\left(\frac{1}{4}-\frac{1}{2}it;\frac{1}{2};-\left(\frac{he^x}{2}\right)^2\right),$$

where ${}_1F_1$ *is the confluent hypergeometric function, then*

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\nabla(h,t,e^{-2x})dt = \pi(e^{x/2}e^{-h^2/(4e^{-2x})} - 2e^{-x/2}\varpi(h,e^{-2x})). \tag{4.17}$$

Our next result generalizes the formula in Corollary 4.1 due to Hardy [Har15, Equation (2)].

Theorem 4.10. *Let* $\beta > 1$ *and* $\phi,\psi \in \clubsuit(\omega,0,\beta)$ *and suppose that* Θ *and* Z *are defined as in* (4.11). *Then we have*

$$\int_0^\infty \frac{\Xi(\frac{t}{2})}{1+t^2}\frac{Z(\frac{1+it}{2})}{\cosh\frac{1}{2}\pi t}dt = -\frac{1}{4}\left\{\sum_{n=1}^{\infty}\left(\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}}dx - \frac{Z(1)}{n}\right) + Z'(1) + \frac{(\gamma-\log 4\pi)}{2}Z(1)\right\}. \tag{4.18}$$

Ferrar's formula in Corollary 4.2, which can be found in [Fer36], is generalized to the following.

Theorem 4.11. *Let* $\beta > 1$ *and* $\phi,\psi \in \clubsuit(\omega,0,\beta)$ *and suppose that* Θ *and* Z *are defined as in* (4.11). *Then we have*

$$4\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(\frac{t}{2})}{1+t^2}Z\left(\frac{1}{2}+it\right)dt = -\pi^{3/2}\left\{2Z'(1) + (\gamma-\log 16\pi)Z(1) + 2\sum_{n=1}^{\infty}\left(\int_0^\infty \frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}dx - \frac{Z(1)}{n}\right)\right\}. \tag{4.19}$$

Corollary 4.2 is proved here using the following lemma, which is interesting in its own right, since it gives an example of a function self-reciprocal in Fourier cosine transform that seems to have been unnoticed before.

Lemma 4.12. *For* $y > 0$, *we have*

$$\int_0^\infty \left(\sum_{n=1}^{\infty} K_0(2\pi nx) - \frac{1}{4x}\right)\cos(2\pi yx)dx = \frac{1}{2}\left(\sum_{n=1}^{\infty} K_0(2\pi ny) - \frac{1}{4y}\right). \tag{4.20}$$

It is interesting to note here that Watson [Wat31, p. 303] proved that for $\mathrm{Re}(\nu) > 0$, the function

$$\frac{1}{2}\Gamma(\nu) + 2\sum_{n=1}^{\infty}\left(nx\sqrt{\frac{\pi}{2}}\right)^{\nu} K_{\nu}(nx\sqrt{2\pi})$$

is self-reciprocal in the generalized Hankel transform of order $2\nu - \frac{1}{2}$.

4.3 The theta transformation formula and proof of Theorem 4.7

Let $\phi, \psi \in \clubsuit(\omega, 0, \alpha)$ for $\omega > 0$ be a pair of cosine reciprocal functions. Suppose that Θ and Z are defined as in (4.11). Let us consider the following integral

$$\mathcal{H}(Z) := \int_0^{\infty} f(t)\Xi(t)Z\left(\frac{1}{2} + it\right)dt, \tag{4.21}$$

where

$$f(t) = g(it)g(-it) \tag{4.22}$$

with g an analytic function of t. By Stirling's formula (A.6), we have

$$\Xi(t) \ll t^{A}e^{-\frac{\pi}{4}t}$$

for a positive constant A. By Lemma 1.9 of Chapter 1 and $\nu = -1/2$ we have

$$Z(\sigma + it) \ll e^{(\frac{\pi}{4} - \omega + \epsilon)t}$$

for every $\epsilon > 0$. Therefore the integral in (4.21) will be convergent as long as $f(t) \ll t^{B}$ for some positive constant B. From Lemma 4.6, we observe $Z(\frac{1}{2} + it)$ is an even function of t. Since $\Xi(t)$ is an even function, (4.21) can be written as

$$\begin{aligned}\mathcal{H}(Z) &= \frac{1}{2}\int_{-\infty}^{\infty} g(it)g(-it)\Xi(t)Z\left(\frac{1}{2} + it\right)dt \\ &= \frac{1}{2i}\int_{(\frac{1}{2})} g\left(s - \frac{1}{2}\right)g\left(\frac{1}{2} - s\right)\xi(s)Z(s)ds\end{aligned} \tag{4.23}$$

by the change $it = s - \frac{1}{2}$. Now let

$$g(s) = \frac{1}{s + \frac{1}{2}}, \quad \text{so that} \quad g\left(s - \frac{1}{2}\right)g\left(\frac{1}{2} - s\right) = \frac{1}{(1-s)s}.$$

This simplifies the expression for $\mathcal{H}$ in (4.23) to

$$\mathcal{H}(Z) = -\frac{1}{4i}\int_{(\frac{1}{2})} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)Z(s)ds.$$

The next step is to move the path of integration from the critical line $\mathrm{Re}(s) = \frac{1}{2}$ past the region of absolute convergence of $\zeta(s)$ at $s = 1 + \varepsilon$ with $\varepsilon > 0$. In doing so, we pick up a contribution of a possible simple pole at $s = 1$ so that

$$\begin{aligned}
\mathcal{H}(Z) &= -\frac{1}{4i}\int_{(1+\varepsilon)} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)Z(s)ds + 2\pi i\frac{1}{4i}\operatorname*{res}_{s=1}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)Z(s) \\
&= -\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{(1+\varepsilon)}\Gamma\left(\frac{s}{2}\right)\left(\pi^{1/2}n\right)^{-s}Z(s)ds + \frac{\pi}{2}Z(1) \\
&= -\frac{\pi}{2}\sum_{n=1}^{\infty}\Theta(\pi^{1/2}n) + \frac{\pi}{2}Z(1).
\end{aligned}$$

By Stirling's formula (A.6) and Lemma 1.9 of Chapter 1 with $\nu = -1/2$ we see that

$$\Gamma\left(\frac{s}{2}\right)Z(s) \ll e^{(-\omega+\epsilon)t} \tag{4.24}$$

for any $\epsilon > 0$. This is enough to justify the vanishing of the integrals along the horizontal segments of the rectangular contour as $t \to \infty$. We have thus shown that

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}Z\left(\frac{1}{2}+it\right)dt = \frac{\pi}{2}Z(1) - \frac{\pi}{2}\sum_{n=1}^{\infty}\Theta(\pi^{1/2}n).$$

We now give a special case of the above identity arising from a specific choice of the cosine reciprocal functions $\phi(x)$ and $\psi(x)$.

4.3.1 Proof of the Dixit integral theorem

Let $\theta, h \in \mathbb{C}$ be fixed parameters with $-\frac{\pi}{2} < \arg\theta < \frac{\pi}{2}$. If we set

$$\phi_h(x,\theta) = \exp(-\theta x^2)\cos(hx),$$

then its cosine reciprocal is then given by

$$\begin{aligned}
\psi_h(x,\theta) &= \frac{2}{\sqrt{\pi}}\int_0^\infty \exp(-\theta u^2)\cos(hu)\cos(2ux)du \\
&= \theta^{-1/2}e^{-h^2/(4\theta)}\exp\left(-\frac{x^2}{\theta}\right)\cosh\left(\frac{hx}{\theta}\right).
\end{aligned}$$

It is clear that $\phi, \psi \in \clubsuit(\omega, 0, \alpha)$. The sum is given by

$$\begin{aligned}
\Theta(x) = \Theta_h(x,\theta) &= \phi_h(x,\theta) + \psi_h(x,\theta) \\
&= \exp(-\theta x^2)\cos(hx) + \theta^{-1/2}e^{-h^2/(4\theta)}\exp\left(-\frac{x^2}{\theta}\right)\cosh\left(\frac{hx}{\theta}\right).
\end{aligned}$$

Then

$$Z_1(s) = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty x^{s-1}\phi_h(x,\theta)dx = \frac{1}{2}\theta^{-s/2}{}_1F_1\left(\frac{s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right)$$

and

$$Z_2(s) = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty x^{s-1}\psi_h(x,\theta)dx = \frac{1}{2}\theta^{(s-1)/2}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right),$$

so that

$$\begin{aligned}Z(s) &= Z_h(s,\theta) = Z_1(s) + Z_2(s)\\ &= \frac{1}{2}\theta^{-s/2}{}_1F_1\left(\frac{s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right) + \frac{1}{2}\theta^{(s-1)/2}{}_1F_1\left(\frac{1-s}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right).\end{aligned}$$

Let us look at the infinite sum

$$\begin{aligned}\Delta(h,\theta) &= \sum_{n=1}^{\infty}\Theta_h(\pi^{1/2}n,\theta)\\ &= \sum_{n=1}^{\infty}\exp(-\pi\theta n^2)\cos(\pi^{1/2}hn) + \theta^{-1/2}e^{-\frac{h^2}{4\theta}}\sum_{n=1}^{\infty}\exp\left(-\frac{\pi n^2}{\theta}\right)\cosh\left(\frac{\pi^{1/2}hn}{\theta}\right).\end{aligned}$$

Define the first infinite sum in the last line by

$$\varpi(h,\theta) := \sum_{n=1}^{\infty}\exp(-\pi\theta n^2)\cos(\pi^{1/2}hn).$$

The functional equation of ϖ is given by [WW62, p. 124, Exercise 18]

$$2\varpi(h,\theta)+1 = \theta^{-1/2}\exp\left(-\frac{h^2}{4\theta}\right)(2\varpi(ih\theta^{-1},\theta^{-1})+1).$$

Going back to Δ, we have

$$\begin{aligned}\Delta(h,\theta) &= \varpi(h,\theta) + \theta^{-1/2}e^{-h^2/(4\theta)}\varpi(ih\theta^{-1},\theta^{-1})\\ &= 2\varpi(h,\theta) + \frac{1}{2} - \frac{1}{2}\theta^{-1/2}e^{-h^2/(4\theta)}.\end{aligned} \tag{4.25}$$

We also note that

$$Z(1) = Z_h(1,\theta) = \frac{1}{2} + \frac{1}{2}\theta^{-1/2}e^{-h^2/(4\theta)} \tag{4.26}$$

and

$$\begin{aligned}Z\left(\frac{1}{2}+it\right) &= Z_h\left(\frac{1}{2}+it,\theta\right)\\ &= \frac{1}{2}\left(\theta^{-\frac{1}{4}-\frac{it}{2}}{}_1F_1\left(\frac{1}{4}+\frac{it}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right) + \theta^{-\frac{1}{4}+\frac{it}{2}}{}_1F_1\left(\frac{1}{4}-\frac{it}{2};\frac{1}{2};-\frac{h^2}{4\theta}\right)\right)\\ &= \frac{1}{2}\theta^{-\frac{1}{4}}\nabla(h,t,\theta).\end{aligned} \tag{4.27}$$

Replacing (4.25),(4.26) and (4.27) in (4.15), and then letting $\theta = e^{-2x}$, we obtain (4.17). This proves Corollary 4.9.

4.3.2 Proof of Corollary 4.8

Just let $h = 0$ in Corollary 4.9.

Remark 4.13. If we substitute $\nu = i\alpha$, with $\alpha \in \mathbb{R}$, in (4.1), write $\cos(i\alpha t) = \frac{1}{2}(e^{-\alpha t}+e^{\alpha t})$ to simplify the integral on the left, and use (4.3), one obtains Lemma 3.2.

Remark 4.14. The choice $(\phi(x),\psi(x)) = (e^{-\theta x}, \frac{2}{\sqrt{\pi}}\frac{\theta}{\theta^2+4x^2}), -\frac{\pi}{2} < \arg\theta < \frac{\pi}{2}$, in (4.15) yields

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} Z\left(\frac{1}{2}+it,\theta\right) dt = -\pi\left(\frac{1}{e^{\sqrt{\pi}\theta}-1} - \frac{1}{\sqrt{\pi}\theta}\right), \tag{4.28}$$

which is a rephrasing of the well-known identity [Tit86, p. 23, Equation (2.7.1)], namely,

$$\int_0^\infty x^{s-1}\left(\frac{1}{e^x-1} - \frac{1}{x}\right)dx = \Gamma(s)\zeta(s), \quad (0 < \mathrm{Re}(s) < 1). \tag{4.29}$$

Equation (4.28) can be obtained from (4.29) using the functional equation for $\zeta(s)$.

4.4 Generalization of Hardy's formula: Proof of Theorem 4.10

Let $g(s) = \frac{1}{4\sqrt{2\pi}}\Gamma(\frac{1}{4}+\frac{s}{2})\Gamma(\frac{-1}{4}+\frac{s}{2})$ in (4.22) so that

$$f(t) = \frac{1}{32\pi^2}\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}+\frac{it}{2}\right)\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)\Gamma\left(\frac{-1}{4}-\frac{it}{2}\right) = \frac{1}{(1+4t^2)\cosh\pi t}$$

and

$$g\left(s-\frac{1}{2}\right)g\left(\frac{1}{2}-s\right) = \frac{1}{4\pi}\Gamma(-s)\Gamma(s-1). \tag{4.30}$$

From (4.21) and (4.23), we have

$$\begin{aligned}\mathcal{H}(Z) &= \frac{1}{8\pi i}\int_{(\frac{1}{2})}\Gamma(-s)\Gamma(s-1)\frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)Z(s)ds \\ &= \frac{-1}{16i}\int_{(\frac{1}{2})}\frac{\zeta(s)}{\sin\pi s}\Gamma\left(\frac{s}{2}\right)Z(s)\pi^{-\frac{s}{2}}ds.\end{aligned} \tag{4.31}$$

Shifting the line of integration to $\mathrm{Re}(s) = 1+\delta$ where $0<\delta<1$, and considering the contribution of the pole of order 2 at $s=1$, we have

$$\begin{aligned}\mathcal{H}(Z) &= \int_0^\infty \frac{\Xi(t)}{(1+4t^2)}\frac{Z(\frac{1}{2}+it)}{\cosh\pi t}dt = \frac{1}{2}\int_0^\infty \frac{\Xi(\frac{t}{2})}{(1+t^2)}\frac{Z(\frac{1+it}{2})}{\cosh\frac{1}{2}\pi t}dt \\ &= \frac{-1}{16i}\left\{\sum_{n=1}^\infty\int_{(1+\delta)}\frac{Z(s)}{\sin\pi s}\Gamma\left(\frac{s}{2}\right)(\sqrt{\pi}n)^{-s}ds - 2\pi i\left(-\frac{Z'(1)}{\pi} - \frac{(\gamma-\log 4\pi)}{2\pi}Z(1)\right)\right\}.\end{aligned} \tag{4.32}$$

Note that

$$|\sin(\pi(\sigma+it))| \geq \frac{e^{\pi|t|}}{2}\left(1-e^{-2\pi|t|}\right) > \frac{e^{\pi|t|}}{4}$$

for large t. Then from (4.24), we find that the integrals along the horizontal segments of the rectangular contour go to 0 as $t\to\infty$. It is well-known [PK01, p. 91, Equation (3.3.10)] that for $0 < \mathrm{Re}(s) < 1$,

$$\frac{1}{2\pi i}\int_{(c)}\frac{x^{-s}}{\sin\pi s}ds = \frac{1}{\pi(1+x)}. \tag{4.33}$$

We employ Parseval's theorem in the form (A.28). Hence, shifting the line of integration to $\mathrm{Re}(s) = c$ with $0 < c < 1$, using (4.12) and the fact that $\phi, \psi \in \clubsuit(\omega, 0, \beta)$, we find that

$$\int_0^\infty \frac{\Xi(\frac{t}{2})}{(1+t^2)} \frac{Z(\frac{1+it}{2})}{\cosh \frac{1}{2}\pi t} dt = -\frac{1}{4}\Bigg\{ \sum_{n=1}^\infty \left(\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}} dx - \frac{Z(1)}{n} \right) + Z'(1) + \frac{(\gamma - \log 4\pi)}{2} Z(1) \Bigg\}, \tag{4.34}$$

which completes the proof of Theorem 4.10.

We now prove Hardy's formula as a special case of the above theorem.

Corollary 4.1. Let $\phi(x) = e^{-a^2x^2}$ in the first equation in (4.8). It is easy to see that $\psi(x) = \frac{1}{a} e^{-x^2/a^2}$, and that $\phi(x)$ and $\psi(x)$ are reciprocal in the Fourier cosine transform. This gives

$$\Theta(x) = e^{-a^2x^2} + \frac{1}{a} e^{-x^2/a^2}.$$

Also one can check that

$$Z(1) = \frac{1}{2}\left(1 + \frac{1}{a}\right), \quad Z'(1) = \frac{\log a}{2}\left(1 - \frac{1}{a}\right), \quad Z\left(\frac{1+it}{2}\right) = \frac{1}{\sqrt{a}} \cos\left(\frac{1}{2} t \log a\right). \tag{4.35}$$

Now

$$\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}} dx - \frac{Z(1)}{n} = \left(\int_0^\infty \frac{e^{-a^2x^2}}{x+n\sqrt{\pi}} dx - \frac{1}{2na} \right) + \frac{1}{a}\left(\int_0^\infty \frac{e^{-x^2/a^2}}{x+n\sqrt{\pi}} dx - \frac{a}{2n} \right) \tag{4.36}$$

$$= -\int_0^\infty e^{-\pi a^2x^2}\left(\frac{1}{n} - \frac{1}{x+n}\right) dx - \frac{1}{a}\int_0^\infty e^{-\pi x^2/a^2}\left(\frac{1}{n} - \frac{1}{x+n}\right) dx,$$

using the fact that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ as indicated in (A.5). Hence,

$$\sum_{n=1}^\infty \left(\int_0^\infty \frac{\Theta(x)}{x+n\sqrt{\pi}} dx - \frac{Z(1)}{n} \right) \tag{4.37}$$

$$= -\int_0^\infty e^{-\pi a^2x^2}(\psi(1+x)+\gamma) dx - \frac{1}{a}\int_0^\infty e^{-\pi x^2/a^2}(\psi(1+x)+\gamma) dx,$$

by the interchange of the order of summation and integration, which is valid because of absolute convergence, and since [Tem96, p. 54, Equation (3.10)]

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{m=0}^\infty \left(\frac{1}{m+x} - \frac{1}{m+1} \right). \tag{4.38}$$

Now we use the integral evaluation

$$\int_0^\infty e^{-\pi a^2x^2}(\gamma + \log x) dx = \frac{\gamma - \log(4\pi a^2)}{4a} \tag{4.39}$$

in (4.37) to write

$$\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{\Theta(x)}{x+n\sqrt{\pi}}dx-\frac{Z(1)}{n}\right) \tag{4.40}$$
$$=-\frac{\gamma-\log(4\pi a^2)}{4a}-\frac{(\gamma-\log(4\pi/a^2))}{4}$$
$$-\int_0^{\infty}e^{-\pi a^2x^2}(\psi(1+x)-\log x)dx-\frac{1}{a}\int_0^{\infty}e^{-\pi x^2/a^2}(\psi(1+x)-\log x)dx.$$

Next we show that

$$\int_0^{\infty}e^{-\pi x^2/a^2}(\psi(1+x)-\log x)dx=a\int_0^{\infty}e^{-\pi a^2x^2}(\psi(1+x)-\log x)dx. \tag{4.41}$$

To that end, note that

$$e^{-\pi x^2/a^2}=2a\int_0^{\infty}e^{-\pi a^2y^2}\cos(2\pi yx)dy. \tag{4.42}$$

Hence,

$$\int_0^{\infty}e^{-\pi x^2/a^2}(\psi(1+x)-\log x)dx=2a\int_0^{\infty}\int_0^{\infty}e^{-\pi a^2y^2}\cos(2\pi yx)$$
$$\times(\psi(1+x)-\log x)dydx \tag{4.43}$$
$$=2a\int_0^{\infty}(\psi(1+x)-\log x)\cos(2\pi yx)dx$$
$$\times\int_0^{\infty}e^{-\pi a^2y^2}dy \tag{4.44}$$

where the interchange of the order of integration can again be justified. From page 220 in Ramanujan's Lost Notebook [Ram88], we see that the function $\psi(1+x)-\log x$ is reciprocal (up to a constant) in the Fourier-cosine transform, namely,

$$\int_0^{\infty}(\psi(1+x)-\log x)\cos(2\pi yx)dx=\frac{1}{2}(\psi(1+y)-\log y). \tag{4.45}$$

This property was later rediscovered by Guinand [Gui47] in 1947.

Substituting (4.45) in (4.43), we obtain (4.41). Now substitute (4.41) in (4.40) to obtain

$$\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{\Theta(x)}{x+n\sqrt{\pi}}dx-\frac{Z(1)}{n}\right) \tag{4.46}$$
$$=-\frac{\gamma-\log(4\pi a^2)}{4a}-\frac{(\gamma-\log(4\pi/a^2))}{4}-2\int_0^{\infty}e^{-\pi a^2x^2}(\psi(1+x)-\log x)dx.$$

Finally from (4.34), (4.35) and (4.46), we obtain Hardy's formula (4.4). □

Remark 4.15. Theorem 1.3 in [Dix13b] can also be obtained as a special case of Theorem 4.10 of this chapter combining the methods in the proofs of Corollaries 4.9 and 4.1.

Remark 4.16. Guinand gave a second proof of this self-reciprocal Fourier cosine transform. For a complete history, see page 293 of [AB13].

4.5 Generalization of Ferrar's formula: Proof of Theorem 4.11

Let $g(s) = \frac{\sqrt{2}}{\frac{1}{2}-s}\Gamma(\frac{1}{4}+\frac{s}{2})$ in (4.22) so that

$$f\left(\frac{t}{2}\right) = g\left(\frac{it}{2}\right)g\left(\frac{-it}{2}\right) = \frac{8}{1+t^2}\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right).$$

Thus from (4.21) and (4.23),

$$\begin{aligned}\mathcal{H}(Z) &= 4\int_0^\infty \Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(t/2)}{1+t^2}Z\left(\frac{1+it}{2}\right)dt \\ &= \frac{1}{2i}\int_{(\frac{1}{2})}\frac{2}{(1-s)s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\xi(s)Z(s)ds \\ &= -\frac{1}{2i}\int_{(\frac{1}{2})}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)Z(s)\pi^{-s/2}ds.\end{aligned} \tag{4.47}$$

We now apply the residue theorem after shifting the line of integration to $\mathrm{Re}(s) = 1+\delta$ where $0<\delta<2$ and considering the contribution of the pole of order 2 at $s=1$. Using (A.6) for the gamma functions $\Gamma(s/2)$ and $\Gamma(1-s/2)$ and (4.24), it is seen that the integrals along the horizontal segments go to zero as $t\to\infty$. This gives

$$\begin{aligned}\mathcal{H}(Z) = -\frac{1}{2i}\Bigg\{&\sum_{n=1}^\infty\int_{(1+\delta)}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)Z(s)(\sqrt{\pi}n)^{-s}ds \\ &- 2\pi i\lim_{s\to1}\frac{d}{ds}\left((s-1)^2\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)Z(s)\pi^{-s/2}\right)\Bigg\} \\ = -\frac{1}{2i}\Bigg\{&-2\pi i\Big(-2\sqrt{\pi}Z'(1)-\sqrt{\pi}(\gamma-\log 16\pi)Z(1)\Big) \\ &+\sqrt{\pi}\sum_{n=1}^\infty\Bigg(\int_{(c)}B\left(\frac{s}{2},\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)Z(s)(\sqrt{\pi}n)^{-s}ds \\ &+\frac{2\pi i}{\sqrt{\pi}}\lim_{s\to1}(s-1)\Gamma\left(\frac{1-s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)Z(s)(\sqrt{\pi}n)^{-s}\Bigg)\Bigg\},\end{aligned} \tag{4.48}$$

where $B(s,z-s)$ is the Euler beta integral given by

$$B(s,z-s) = \int_0^\infty\frac{x^{s-1}}{(1+x)^z}dx = \frac{\Gamma(s)\Gamma(z-s)}{\Gamma(z)}, \quad 0<\mathrm{Re}(s)<\mathrm{Re}(z). \tag{4.49}$$

From (4.49), we have for $0<c=\mathrm{Re}(s)<1$,

$$\frac{1}{2\pi i}\int_{(c)}B\left(\frac{s}{2},\frac{1-s}{2}\right)x^{-s}ds = \frac{2}{\sqrt{1+x^2}}. \tag{4.50}$$

Along with (4.12), (A.28), and the fact that $\phi,\psi\in\clubsuit(\omega,0,\beta)$, this gives

$$\mathcal{H}(Z) = 4\int_0^\infty\Gamma\left(\frac{1+it}{4}\right)\Gamma\left(\frac{1-it}{4}\right)\frac{\Xi(t/2)}{1+t^2}Z\left(\frac{1}{2}+it\right)dt \tag{4.51}$$

$$= -\pi^{3/2}\left\{2Z'(1) + (\gamma - \log 16\pi)Z(1) + 2\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}dx - \frac{Z(1)}{n}\right)\right\}.$$

Now we prove Ferrar's formula as a special case of the above theorem.

Corollary 4.2. As in the proof of Corollary 4.1, we consider the pair of reciprocal functions $(\phi(x), \psi(x)) = (e^{-a^2x^2}, \frac{1}{a}e^{-x^2/a^2})$. From (4.35), we have

$$\begin{aligned}\int_0^{\infty}\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}dx - \frac{Z(1)}{n} &= \left(\int_0^{\infty}\frac{e^{-a^2x^2}}{\sqrt{x^2+\pi n^2}}dx - \frac{1}{2na}\right)\\ &\quad + \frac{1}{a}\left(\int_0^{\infty}\frac{e^{-x^2/a^2}}{\sqrt{x^2+\pi n^2}}dx - \frac{a}{2n}\right) \qquad (4.52)\\ &= \int_0^{\infty}e^{-\frac{a^2t^2}{4\pi}}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}} - \frac{1}{2\pi n}\right)dt\\ &\quad + \frac{1}{a}\int_0^{\infty}e^{-\frac{t^2}{4\pi a^2}}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}} - \frac{1}{2\pi n}\right)dt \qquad (4.53)\end{aligned}$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\int_0^{\infty}\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}dx - \frac{Z(1)}{n}\right) &= \int_0^{\infty}e^{-\frac{a^2t^2}{4\pi}}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}} - \frac{1}{2\pi n}\right)dt\\ &\quad + \frac{1}{a}\int_0^{\infty}e^{-\frac{t^2}{4\pi a^2}}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}} - \frac{1}{2\pi n}\right)dt\\ &=: J(a) + \frac{1}{a}J\left(\frac{1}{a}\right), \qquad (4.54)\end{aligned}$$

say. Here the interchange of the order of summation and integration is justified by absolute convergence. From [Wat31, Equation 6], we have, for $\mathrm{Re}(t) > 0$,

$$2\sum_{n=1}^{\infty}K_0(nt) = \pi\left\{\frac{1}{t} + 2\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{t^2+4\pi^2n^2}} - \frac{1}{2n\pi}\right)\right\} + \gamma + \log\left(\frac{t}{2}\right) - \log 2\pi. \qquad (4.55)$$

Hence,

$$\begin{aligned}J(a) &= \int_0^{\infty}e^{-\frac{a^2t^2}{4\pi}}\left(\frac{1}{2\pi}\left(-\gamma + \log 4\pi - \log t + 2\sum_{n=1}^{\infty}K_0(nt)\right) - \frac{1}{2t}\right)dt \qquad (4.56)\\ &= \frac{(-\gamma+\log 4\pi)}{2\pi}\frac{\pi}{a} - \frac{1}{2\pi}\left(-\frac{\pi}{2a}(\gamma - \log\pi + 2\log a)\right)\\ &\quad + \frac{1}{\pi}\int_0^{\infty}e^{-\frac{a^2t^2}{4\pi}}\left(\sum_{n=1}^{\infty}K_0(nt) - \frac{\pi}{2t}\right)dt \qquad (4.57)\\ &= -\frac{\gamma}{4a} + \frac{\log(4\sqrt{\pi}a)}{2a} + \frac{1}{\pi}\int_0^{\infty}e^{-\frac{a^2t^2}{4\pi}}\left(\sum_{n=1}^{\infty}K_0(nt) - \frac{\pi}{2t}\right)dt.\end{aligned}$$

Thus

$$J\left(\frac{1}{a}\right) = -\frac{\gamma a}{4} + \frac{a\log(\frac{4\sqrt{\pi}}{a})}{2} + \frac{1}{\pi}\int_0^{\infty}e^{-\frac{t^2}{4\pi a^2}}\left(\sum_{n=1}^{\infty}K_0(nt) - \frac{\pi}{2t}\right)dt. \qquad (4.58)$$

Next we prove that

$$\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)dt = a\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)dt. \tag{4.59}$$

Using (4.42), we have

$$\begin{aligned}\int_0^\infty e^{-\frac{t^2}{4\pi a^2}}\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)dt &= \int_0^\infty \left(2a\int_0^\infty e^{-\pi a^2y^2}\cos(yt)dy\right)\\ &\qquad\times\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)dt \qquad (4.60)\\ &= 2a\int_0^\infty e^{-\pi a^2y^2}dy\int_0^\infty\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)\cos(yt)dt.\end{aligned}$$

Now letting $x=t/(2\pi)$ in Lemma 4.12 below, using the resulting identity in the above equation and then again employing a change of variable $y=t/(2\pi)$, we obtain (4.59). Thus from (4.54), (4.56), (4.58) and (4.59), we deduce that

$$\begin{aligned}&\sum_{n=1}^\infty\left(\int_0^\infty\frac{\Theta(x)}{\sqrt{x^2+\pi n^2}}dx-\frac{Z(1)}{n}\right) \qquad (4.61)\\ &= -\frac{\gamma}{4a}+\frac{\log(4\sqrt{\pi}a)}{2a}-\frac{\gamma}{4}+\frac{\log(\frac{4\sqrt{\pi}}{a})}{2}+\frac{2}{\pi}\int_0^\infty e^{-\frac{a^2t^2}{4\pi}}\left(\sum_{n=1}^\infty K_0(nt)-\frac{\pi}{2t}\right)dt.\end{aligned}$$

Finally, from (4.35), (4.51) and (4.61), we obtain Ferrar's formula (4.5). □

We conclude this section with the proof of Lemma 4.12.

Lemma 4.12. [] We first show that the Mellin transform of $\sum_{n=1}^\infty K_0(2\pi nx)-\frac{1}{4x}$ for $0<c=\text{Re } s<1$ is given by

$$\int_0^\infty x^{s-1}\left(\sum_{n=1}^\infty K_0(2\pi nx)-\frac{1}{4x}\right)dx = \frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \tag{4.62}$$

Observe that from (4.55), we have

$$\sum_{n=1}^\infty K_0(2\pi nx)-\frac{1}{4x}=\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{x}{2}\right)+\frac{1}{2}\sum_{n=1}^\infty\left(\frac{1}{\sqrt{x^2+n^2}}-\frac{1}{n}\right). \tag{4.63}$$

From (4.50), for $0<c=\text{Re}(s)<1$, we have

$$\frac{1}{2\pi i}\int_{(c)}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)x^{-s}ds=\frac{1}{\sqrt{1+x^2}}. \tag{4.64}$$

Shifting the line of integration to $c'=\text{Re}(s)>1$, we get

$$\frac{1}{2\pi i}\int_{(c')}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)x^{-s}ds=\frac{1}{\sqrt{1+x^2}}+\lim_{s\to1}\frac{(s-1)}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)x^{-s} \tag{4.65}$$

$$= \frac{1}{\sqrt{1+x^2}} - \frac{1}{x}.$$

Thus

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{1+(\frac{n}{x})^2}} - \frac{1}{n/x}\right) = \sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{(c')}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\left(\frac{n}{x}\right)^{-s}ds \tag{4.66}$$
$$= \frac{1}{2\pi i}\int_{(c')}\frac{1}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^s ds.$$

Shifting the line of integration back to $0 < c = \mathrm{Re}(s) < 1$, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{1+(\frac{n}{x})^2}} - \frac{1}{n/x}\right) &= \frac{1}{2\pi i}\int_{(c)}\frac{x^s}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)ds \\ &\quad + \lim_{s\to 1}\frac{d}{ds}\left(\frac{(s-1)^2}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^s\right) \\ &= \frac{1}{2\pi i}\int_{(c)}\frac{x^s}{2\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)ds - x\left(\gamma + \log\left(\frac{x}{2}\right)\right),\end{aligned} \tag{4.67}$$

so that

$$\frac{\gamma}{2} + \frac{1}{2}\log\left(\frac{x}{2}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{x^2+n^2}} - \frac{1}{n}\right) = \frac{1}{2\pi i}\int_{(c)}\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)x^{s-1}ds. \tag{4.68}$$

Now replace s by $1-s$ in the above equation and then use (4.63) to complete the proof of (4.62). Now note that for $0 < c = \mathrm{Re}\ s < 1$,

$$\int_0^{\infty} x^{s-1}\cos(2\pi yx)dx = (2\pi y)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s). \tag{4.69}$$

Now let $f(x) = \sum_{n=1}^{\infty} K_0(2\pi nx) - \frac{1}{4x}$ and $g(x) = \cos(2\pi yx)$ in Parseval's formula (A.29). Then from (4.62), (4.69) and (A.29), for $0 < c = \mathrm{Re}\ s < 1$, we have

$$\int_0^{\infty}\left(\sum_{n=1}^{\infty}K_0(2\pi nx) - \frac{1}{4x}\right)\cos(2\pi yx)dx \tag{4.70}$$
$$= \frac{1}{2\pi i}\int_{(c)}\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(s)(2\pi y)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)ds.$$

Using the functional equation for $\zeta(s)$ in the form $\zeta(1-s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\cos(\frac{1}{2}\pi s)$ in the above equation and employing (4.62), we finally obtain (4.20). This completes the proof of the Lemma. □

Remark 4.17. Theorem 1.4 in [Dix13b] can also be obtained as a special case of Theorem 4.11 of this chapter combining the methods in the proofs of Corollaries 4.9 and 4.2.

4.6 Two powers of Ξ: Koshliakov and Ramanujan

In the previous section the transformation relating the functions φ and ψ was a Fourier cosine transform. In our present circumstances, the kernel that needs to be used was introduced by G. H. Hardy. Specifically, let $\varphi(x)$ and $\psi(x)$ be related by the

$$\varphi(x) = 2\int_0^\infty \psi(t)\left(\frac{2}{\pi}K_0(4\sqrt{tx}) - Y_0(4\sqrt{tx})\right)dt, \tag{4.71}$$

and

$$\psi(x) = 2\int_0^\infty \varphi(t)\left(\frac{2}{\pi}K_0(4\sqrt{tx}) - Y_0(4\sqrt{tx})\right)dt. \tag{4.72}$$

where $Y_0(z)$ denotes the Bessel Y function. The kernel appearing in the integral transforms (4.71) and (4.72) will be called the Koshliakov kernel. One can see that if φ and ψ are reciprocal under the Koshliakov kernel, then by making the change $x \to x\theta$ with $\theta > 0$, one has that $\varphi(\theta x)$ and $\theta^{-1}\psi(\theta^{-1}x)$ are also reciprocal under the Koshliakov kernel.

Definition 4.18. Let $0 < \omega \leq \pi$ and $0 < \eta$. If $f(z)$ is such that

i) $f(z)$ is analytic with $z = re^{i\theta}$, regular in the angle defined by $r > 0$, $|\theta| < \omega$,

ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O_\delta(|z|^{-\delta}) & \text{if } |z| \leq 1, \\ O(|z|^{-\eta-1}) & \text{if } |z| > 1, \end{cases} \tag{4.73}$$

for every positive δ and uniformly in any angle $|\theta| < \omega$, then we say that f belongs to the class $\spadesuit(\eta, \omega)$ and write $f(z) \in \spadesuit(\eta, \omega)$.

Let us next define $Z_1(s)$ and $Z_2(s)$ in terms of the Mellin transforms of φ and ψ normalized by $\Gamma^2(s/2)$,

$$\Gamma^2\left(\frac{s}{2}\right)Z_1(s) := \int_0^\infty \varphi(x)x^{s-1}dx \quad \text{and} \quad \Gamma^2\left(\frac{s}{2}\right)Z_2(s) := \int_0^\infty \psi(x)x^{s-1}dx, \tag{4.74}$$

each valid in a specific vertical strip in the complex s-plane. Much like in the previous section where we had one single power of $\Gamma(\frac{s}{2})$, in case of a non-empty intersection of the two corresponding vertical strips, the Mellin inversion theorem gives

$$\varphi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma^2\left(\frac{s}{2}\right)Z_1(s)x^{-s}ds \quad \text{and} \quad \psi(x) = \frac{1}{2\pi i}\int_{(c)} \Gamma^2\left(\frac{s}{2}\right)Z_2(s)x^{-s}ds,$$

where $\mathrm{Re}(s) = c$ lies in the intersection. Moreover, let us define

$$\Theta(x) := \varphi(x) + \psi(x) \quad \text{and} \quad Z(s) := Z_1(s) + Z_2(s) \tag{4.75}$$

so that

$$\Gamma^2\left(\frac{s}{2}\right)Z(s) = \int_0^\infty x^{s-1}\Theta(x)dx \tag{4.76}$$

for values of s in the intersection of the two strips. We will later show that $Z(\frac{1}{2}+it)$ is a real and even function of t when t is real.

The two main results of integrals involving $\Xi^2(t)$ are stated below.

Theorem 4.19 (Generalized Koshliakov integral theorem)**.** *Suppose* $\varphi, \psi \in \spadesuit(\eta,\omega)$ *is a pair of Koshliakov reciprocal functions. One has*

$$16\int_0^\infty \Xi^2\left(\frac{t}{2}\right) Z\left(\frac{1+it}{2}\right)\frac{dt}{(1+t^2)^2} = \frac{\pi}{2}\left(\sum_{n=1}^\infty d(n)\Theta(\pi n) - (Z'(1)+(\gamma-\log 4\pi)Z(1))\right), \tag{4.77}$$

where Z *and* Θ *are defined by* (4.75)*.*

Theorem 4.20 (Generalized Ramanujan integral theorem)**.** *For* $\varphi, \psi \in \spadesuit(\eta,\omega)$ *one has*

$$\begin{aligned} &-\frac{2}{\pi^{3/2}}\int_0^\infty \left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \Xi^2\left(\frac{t}{2}\right) Z\left(\frac{1+it}{2}\right)\frac{dt}{1+t^2} \\ &= (\gamma-\log 2\pi)Z(1) + Z'(1) - \pi\sum_{n=1}^\infty nd(n)\int_0^\infty \frac{\Theta(x)x}{(x^2+\pi^2n^2)^{3/2}}dx \end{aligned}$$

where Θ *and* Z *are given by* (4.75)*.*

For any given $\eta > 0$, the function $K_0(x)$ is in $\spadesuit(\eta,\omega)$. It is known that $K_0(2x)$ is self-reciprocal under the Koshliakov transform. Thus the pair

$$\varphi(x) = \psi(x) = K_0(2x), \tag{4.78}$$

satisifes (4.71) and (4.72). If one takes

$$\mathcal{L}(x) := \frac{1}{\pi^{3/2}}\{e^x \operatorname{li}(e^{-x}) + e^{-x}\operatorname{li}(e^x)\}, \tag{4.79}$$

where $\operatorname{li}(x)$ denotes the logarithmic integral function, i.e. the Cauchy principal value

$$\operatorname{li}(x) = \lim_{\varepsilon\to 0^+}\left(\int_0^{1-\varepsilon}\frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right), \tag{4.80}$$

then it follows that $\mathcal{L} \in \spadesuit(\eta,\omega)$ for any $0<\eta<1/2$.

$$\varphi(x) = \exp(-x) \quad \text{and} \quad \psi(x) = -2\sqrt{\pi}\mathcal{L}(4x), \tag{4.81}$$

is in $\spadesuit(\eta,\omega)$ for $0<\eta<1/2$. Moreover, Dixon and Ferrar showed [DF34, DF35] that φ and ψ are Koshliakov reciprocal. Thus the pair (4.81) also satisifes (4.71) and (4.72).

As a special case when we choose (4.78) in Theorem 4.19, we obtain Corollary 4.3. Another special case coming from (4.81) yields a new result not seen in the literature.

Corollary 4.21. *For* $\theta > 0$ *one has*

$$\begin{aligned} \frac{32}{\pi}\int_0^\infty \Xi^2\left(\frac{t}{2}\right)\mathcal{F}_\theta(t)\frac{dt}{(1+t^2)^2} &= \sum_{n=1}^\infty d(n)\mathcal{A}_\theta(\pi n) \\ &\quad - \left(\frac{\log 256 - 4\log\theta - \pi\theta}{4\pi\theta} + \frac{\gamma-\log 4\pi}{\pi\theta}\right) \end{aligned} \tag{4.82}$$

where

$$\mathcal{F}_\theta(t) := \frac{\theta^{-1/2-it/2}\Gamma(\frac{1}{2}+\frac{1}{2}it)}{\Gamma^2(\frac{1}{4}+\frac{1}{4}it)} + \frac{2^{-it}\theta^{-1/2+it/2}\cot(\frac{\pi}{4}+\frac{\pi}{4}it)\Gamma(\frac{1}{2}+\frac{1}{2}it)}{\Gamma^2(\frac{1}{4}+\frac{1}{4}it)} \tag{4.83}$$

and

$$\mathcal{A}_\theta(x) = \exp(-\theta x) - \frac{2\sqrt{\pi}}{\theta}\mathcal{L}\left(\frac{4x}{\theta}\right). \tag{4.84}$$

Likewise, a very special case of Theorem 4.20 can be found in [Ram15, eq. (22)] and [Ram88, p. 220]. See also [Dix13c] for further details. This special case follows by taking the pair (4.78).

4.7 Bolts and nuts

Let us start with the following auxiliary result.

Lemma 4.22. *Suppose that* $\varphi, \psi \in \spadesuit(\eta,\omega)$ *is a pair of Koshliakov reciprocal functions. One has*

1. $Z(s) = Z(1-s)$ *for all* s *in* $-\eta < \mathrm{Re}(s) < 1+\eta$,
2. $Z(\sigma+it) \ll e^{(\frac{\pi}{2}-\omega+\varepsilon)|t|}$ *for every* $\varepsilon > 0$.

Proof. Fix $\eta > 0$, $0 < \omega \leq \pi$ and let $\varphi, \psi \in \spadesuit(\eta,\omega)$. The first part of the claim is the functional equation between Z_1 and Z_2. To prove this, note that

$$\begin{aligned}
\Gamma^2\left(\frac{1-s}{2}\right)Z_1(1-s) &= \int_0^\infty \varphi(x)x^{-s}dx \\
&= 2\int_0^\infty x^{-s}\int_0^\infty \psi(t)\left(\frac{2}{\pi}K_0(4\sqrt{tx}) - Y_0(4\sqrt{tx})\right)dtdx \\
&= 2\int_0^\infty \psi(t)\int_0^\infty x^{-s}\left(\frac{2}{\pi}K_0(4\sqrt{tx}) - Y_0(4\sqrt{tx})\right)dxdt \\
&= 2\pi^{2-2s}\int_0^\infty \psi(t)t^{s-1}\int_0^\infty u^{-s}\left(\frac{2}{\pi}K_0(4\pi\sqrt{u}) - Y_0(4\pi\sqrt{u})\right)dudt,
\end{aligned}$$

provided the interchange of the order of integration is valid and where we have used the substitution $u = tx/\pi^2$. Next by Lemma 5.1 from [DM15] with $y = 1$, $z = 0$ and s replaced by $1-s$ and for

$$\frac{1}{4} < \mathrm{Re}(s) < 1$$

we have

$$\int_0^\infty u^{-s}\left(\frac{2}{\pi}K_0(4\pi\sqrt{u}) - Y_0(4\pi\sqrt{u})\right)du = 4^{s-1}\pi^{2s-3}(1-\cos(\pi s))\Gamma^2(1-s).$$

Therefore, when we insert this in the above, we obtain

$$\Gamma^2\left(\frac{1-s}{2}\right)Z_1(1-s) = \frac{2^2\pi^{2-2s}}{2^{2-2s}\pi^{3-2s}}\Gamma^2(1-s)\sin^2\left(\frac{\pi s}{2}\right)\int_0^\infty \psi(t)t^{s-1}dt$$

$$= \frac{2^{2s}}{\pi}\Gamma^2(1-s)\sin^2\left(\frac{\pi s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)Z_2(s).$$

Thus, we end up with

$$Z_1(1-s) = \frac{2^{2s}}{\pi}\frac{\Gamma^2(1-s)}{\Gamma^2(\frac{1-s}{2})}\sin^2\left(\frac{\pi s}{2}\right)\Gamma^2\left(\frac{s}{2}\right)Z_2(s) = Z_2(s). \tag{4.85}$$

To prove that the interchange is valid by Fubini's theorem, we reason as follows. First, assume that $\frac{3}{4} < \mathrm{Re}(s) < 1$ and fix $\varepsilon_0 > 0$ such that

$$\frac{3}{4} + \varepsilon_0 \leqslant \mathrm{Re}(s) \leqslant 1 - \varepsilon_0.$$

Then

$$|x^{-s}| \leq \begin{cases} x^{\varepsilon_0 - 1}, & \text{if } 0 \leq x \leq 1, \\ x^{-\varepsilon_0 - 3/4}, & \text{if } x \geq 1. \end{cases} \tag{4.86}$$

By the asymptotics of the functions K_0 and Y_0 we have

$$|K_0(v) - Y_0(v)| \ll \begin{cases} 1 + |\log v|, & \text{if } 0 \leq v \leq 1, \\ v^{-1/2}, & \text{if } v \geq 1. \end{cases} \tag{4.87}$$

so that

$$|K_0(\sqrt{tx}) - Y_0(\sqrt{tx})| \ll \begin{cases} 1 + |\log(tx)|, & \text{if } 0 \leq xt \leq 1, \\ t^{-1/4}x^{-1/4}, & \text{if } tx \geq 1. \end{cases} \tag{4.88}$$

We now divide the first quadrant of the t, x plane into six different regions whose boundaries are determined by the $t = 1$, $x = 1$ and the hyperbola $xt = 1$. We set

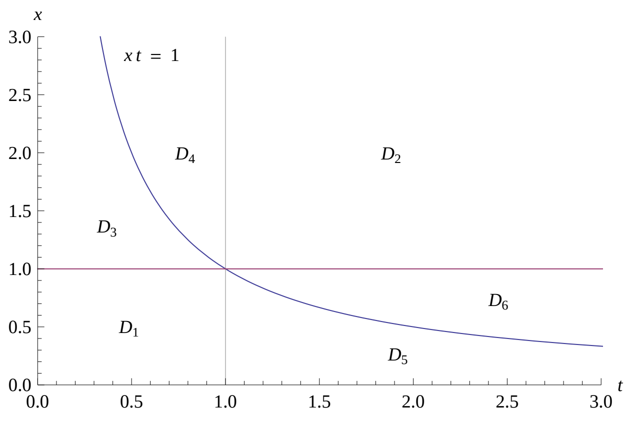

FIGURE 4.1: Regions from the hyperbola $xt = 1$.

$$F(t,x) := x^{-s}\psi(t)(K_0(\sqrt{tx}) - Y_0(\sqrt{tx})),$$

and we examine the double integral

$$I_{s,\psi} := I_1 + I_2 + \cdots + I_6, \quad \text{with} \quad I_j := \iint_{\mathcal{D}_j} |F(x,t)| d\lambda,$$

where $d\lambda$ denotes the Lebesgue measure. We estimate each I_j separately. Let us assume that $\varepsilon < \varepsilon_0$. For the first region, we have

$$\begin{aligned} I_1 &\ll \iint_{D_1} \frac{1}{x^{1-\varepsilon_0}} \frac{1}{t^\delta}(1+|\log tx|)dxdt \ll_\varepsilon \iint_{D_1} \frac{1}{x^{1-\varepsilon_0}} \frac{1}{t^\delta} \frac{1}{(xt)^\varepsilon} dxdt \\ &= \left(\int_0^1 \frac{dx}{x^{1-\varepsilon_0+\varepsilon}}\right)\left(\int_0^1 \frac{dt}{t^{\delta+\varepsilon}}\right) < \infty \end{aligned}$$

For the second region, we have

$$I_2 \ll \iint_{D_2} \frac{1}{x^{3/4+\varepsilon_0}} \frac{1}{t^{1+\eta}} \frac{1}{t^{1/4}x^{1/4}} dxdt = \left(\int_1^\infty \frac{dx}{x^{1+\varepsilon_0}}\right)\left(\int_1^\infty \frac{tx}{t^{5/4+\eta}}\right) = \frac{1}{\varepsilon_0}\frac{4}{1+4\eta} < \infty.$$

For the third region

$$\begin{aligned} I_3 &\ll \iint_{D_3} \frac{1}{x^{3/4+\varepsilon_0}} \frac{1}{t^\delta}(1+\log|tx|)dxdt \ll_\varepsilon \iint_{D_3} \frac{1}{x^{3/4+\varepsilon_0+\varepsilon}} \frac{1}{t^{\delta+\varepsilon}} dxdt \\ &= \int_1^\infty \frac{1}{x^{3/4+\varepsilon_0+\varepsilon}} \left(\int_0^{1/x} \frac{dt}{t^{\delta+\varepsilon}}\right) dx \ll \int_1^\infty \frac{dx}{x^{7/4-\delta+\varepsilon_0}} < \infty. \end{aligned}$$

The double integral over the fourth region yields

$$I_4 \ll \iint_{D_4} \frac{1}{x^{3/4+\varepsilon_0}} \frac{1}{t^\delta} \frac{1}{t^{1/4}x^{1/4}} dxdt = \int_1^\infty \frac{1}{x^{1+\varepsilon_0}} \int_{1/x}^1 \frac{dt}{t^{\delta+1/4}} dx \ll \int_1^\infty \frac{1}{x^{1+\varepsilon_0}} dx < \infty.$$

For the fifth region, we have

$$\begin{aligned} I_5 &\ll \iint_{D_5} \frac{1}{x^{1-\varepsilon_0}} \frac{1}{t^{1+\eta}}(1+\log|tx|)dxdt \ll_\varepsilon \iint_{D_5} \frac{1}{x^{1-\varepsilon_0+\varepsilon}} \frac{1}{t^{1+\eta+\varepsilon}} dxdt \\ &= \int_1^\infty \frac{1}{t^{1+\eta+\varepsilon}} \left(\int_0^{1/t} \frac{dx}{x^{1-\varepsilon_0+\varepsilon}}\right) dt \ll_{\varepsilon,\varepsilon_0} \int_1^\infty \frac{dt}{t^{1+\eta+\varepsilon_0}} = \frac{1}{\eta+\varepsilon_0} < \infty. \end{aligned}$$

Finally,

$$\begin{aligned} I_6 &\ll \iint_{D_6} \frac{1}{x^{1-\varepsilon_0}} \frac{1}{t^{1+\eta}} \frac{1}{t^{1/4}x^{1/4}} dxdt = \int_1^\infty \frac{1}{t^{5/4+\eta}} \int_{1/t}^1 \frac{dx}{x^{5/4-\varepsilon_0}} dt \ll \int_1^\infty \frac{1}{t^{1+\eta+\varepsilon_0}} dt \\ &= \frac{1}{\eta+\varepsilon_0} < \infty. \end{aligned} \tag{4.89}$$

Now let us look at $Z_2(s)$ and the Mellin transform of $\psi(t)$. We split up the integral as

$$\int_0^\infty |\psi(t)||t^{s-1}|dt = \int_0^1 |\psi(t)||t^{s-1}|dt + \int_1^\infty |\psi(t)||t^{s-1}|dt.$$

Since $\varphi \in \spadesuit(\eta, \omega)$, for the latter integral, we have

$$\int_1^\infty |\psi(t)||t^{s-1}|dt \ll \int_1^\infty t^{\mathrm{Re}(s)-2-\eta} dt < \infty$$

provided that $\mathrm{Re}(s) < 1+\eta$. Similarly, for the first integral, we have

$$\int_0^1 |\psi(t)|t^{\mathrm{Re}(s)-1} dt \ll \int_0^1 t^{\mathrm{Re}(s)-1-\delta} dt < \infty$$

provided that $\mathrm{Re}(s) > \delta$. This shows that the function $Z_2(s)$ is well-defined and analytic for s in the region

$$\delta < \mathrm{Re}(s) < 1 + \eta, \tag{4.90}$$

for every $\delta > 0$, and similarly for $Z_1(s)$. Thus, by analytic continuation the equality $Z_1(1-s) = Z_2(s)$ holds in the vertical strip

$$-\eta < \mathrm{Re}(s) < 1 + \eta.$$

To prove the second part of the lemma, let us consider the line along any radius vector r and angle θ (we would choose $-\theta$, if $t = \mathrm{Im}(s) < 0$), where $|\theta| < \omega$. Then by Cauchy's theorem we can deform the integral (4.74) to

$$\Gamma^2\left(\frac{\sigma + it}{2}\right) Z_1(\sigma + it) = \int_0^\infty r^{\sigma+it-1} e^{i\theta(\sigma+it)} \varphi(re^{i\theta})\, dr,$$

where $\theta, t > 0$. Therefore, by splitting the range of integration to $[0, 1]$ and $[1, \infty]$ and the fact that Z_1 is analytic in the region defined by (4.90) we see that

$$\left|\Gamma^2\left(\frac{\sigma + it}{2}\right) Z_1(\sigma + it)\right| \leq e^{-\theta t} \int_0^\infty r^{\sigma-1} |\varphi(re^{i\theta})| dr \ll e^{-|\theta||t|}, \tag{4.91}$$

since $\varphi \in \spadesuit(\eta, \omega)$. Now combining (4.91) and (A.6) we get

$$Z_1(1-s) = Z_2(s) \ll e^{(\frac{\pi}{2} - |\theta|)|t|} \ll e^{(\frac{\pi}{2} - \omega + \varepsilon)|t|}, \tag{4.92}$$

for every positive ε. This completes the proof the lemma. $\square$

4.8 The Koshliakov case

4.8.1 Proof of Theorem 4.19

Let us consider

$$\mathcal{K}(Z) = \int_0^\infty f\left(\frac{t}{2}\right) \Xi^2\left(\frac{t}{2}\right) Z\left(\frac{1 + it}{2}\right) dt$$

where $f(t) = g(it)g(-it)$ with g an analytic function of t. Then one has

$$\mathcal{K}(Z) = \frac{1}{i} \int_{(\frac{1}{2})} g\left(s - \frac{1}{2}\right) g\left(\frac{1}{2} - s\right) \xi^2(s) Z(s) ds.$$

Choose

$$g(s) = \left(s + \frac{1}{2}\right)^{-2} \quad \text{so that} \quad g\left(s - \frac{1}{2}\right) g\left(\frac{1}{2} - s\right) = \frac{1}{s^2(s-1)^2}.$$

This simplifies the above expression for K to

$$\mathcal{K}(Z) = \frac{1}{4i} \int_{(\frac{1}{2})} \pi^{-s} \Gamma^2\left(\frac{s}{2}\right) \zeta^2(s) Z(s) ds.$$

The next step is to move the path of integration from the critical line $\mathrm{Re}(s) = \frac{1}{2}$ past the region of absolute convergence of $\zeta^2(s)$ at $s = 1+\varepsilon$ with $\varepsilon > 0$. In doing so, we pick up a contribution of a double pole at $s = 1$ so that

$$\begin{aligned}\mathcal{K}(Z) &= \frac{1}{4i}\left[\int_{(1+\varepsilon)} \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)Z(s)ds - 2\pi i \operatorname*{res}_{s=1} \pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)Z(s)\right] \\ &= \frac{1}{4i}\left[\sum_{n=1}^{\infty} d(n)\int_{(1+\varepsilon)} (\pi n)^{-s}\Gamma^2\left(\frac{s}{2}\right)Z(s)ds - 2\pi i(Z'(1) + (\gamma - \log 4\pi)Z(1))\right] \\ &= \frac{\pi}{2}\sum_{n=1}^{\infty} d(n)\Theta(\pi n) - \frac{\pi}{2}(Z'(1) + (\gamma - \log 4\pi)Z(1)),\end{aligned}$$

since

$$\zeta^2(s) = \sum_{n=1}^{\infty}\frac{d(n)}{n^s} \tag{4.93}$$

for $\mathrm{Re}(s) > 1$. By Stirling's formula (A.6) and Lemma 4.22, we obtain

$$\Gamma^2\left(\frac{s}{2}\right)Z(s) \ll e^{-(\omega+\varepsilon)t} \tag{4.94}$$

for any $\varepsilon > 0$. This justifies the vanishing of the integrals along horizontal segments of the rectangular contour as $t \to \infty$.

4.8.2 Proof of Corollary 4.3

Suppose $\theta > 0$. Let $\nu = 0$ and replace t by $2t\theta$ and x by $2x$ in equation (5.6) of [DM15] to obtain

$$2\int_0^{\infty} K_0(2t\theta)\left(\frac{2}{\pi}K_0(4\sqrt{tx}) - Y_0(4\sqrt{tx})\right)dt = \frac{1}{\theta}K_0\left(\frac{2x}{\theta}\right).$$

Likewise, integrating $\theta^{-1}K_0(2t/\theta)\lambda_0(4\sqrt{tx})$ yields $K_0(2\theta x)$. This means that we can choose the pair

$$\varphi(x) = K_0(2\theta x) \quad \text{and} \quad \psi(x) = \frac{1}{\theta}K_0\left(\frac{2x}{\theta}\right).$$

Now for $\mathrm{Re}(s) > \pm\mathrm{Re}(z)$ and $a > 0$, we have (from equation 5.10 of [DM15])

$$\int_0^{\infty} x^{s-1}K_z(ax)dx = 2^{s-2}a^{-s}\Gamma\left(\frac{s-z}{2}\right)\Gamma\left(\frac{s+z}{2}\right).$$

Consequently, from the definitions of Z_1 and Z_2, we arrive at

$$Z_1(s) = \frac{\theta^{-s}}{4}, \quad \text{and} \quad Z_2(s) = \frac{\theta^{s-1}}{4}.$$

Hence, we have

$$Z(1) = \frac{1}{4\theta} + \frac{1}{4} \quad \text{and} \quad Z'(1) = \frac{1}{4}\log\theta - \frac{1}{4\theta}\log\theta,$$

as well as

$$Z\left(\frac{1+it}{2}\right)=\frac{\theta^{-1/2-it/2}}{4}+\frac{\theta^{-1/2+it/2}}{4}=\frac{1}{2\sqrt{\theta}}\cos\left(\frac{1}{2}t\log\theta\right).$$

Therefore, inserting these results into Theorem 4.19 we obtain

$$\begin{aligned}\mathcal{K}(Z) &:= \frac{8}{\sqrt{\theta}}\int_0^\infty \frac{\Xi^2(\frac{t}{2})}{(1+t^2)^2}\cos\left(\frac{1}{2}t\log\theta\right)dt\\ &= \frac{\pi}{2}\left\{\sum_{n=1}^\infty d(n)\left(K_0(2\pi n\theta)+\frac{1}{\theta}K_0\left(\frac{2\pi n}{\theta}\right)\right)-\left[\frac{\log\theta}{4}\frac{\theta-1}{\theta}+\frac{\gamma-\log 4\pi}{4}\frac{\theta+1}{\theta}\right]\right\}\\ &= \frac{\pi}{2}\left\{\sum_{n=1}^\infty d(n)K_0(2\pi n\theta)-\frac{\gamma-\log 4\pi}{4\theta}+\frac{1}{\theta}\sum_{n=1}^\infty d(n)K_0\left(\frac{2\pi n}{\theta}\right)-\frac{\gamma-\log\frac{4\pi}{\theta}}{4}\right\}.\end{aligned}$$

Now note that from Koshliakov's transformation formula [Kos29]

$$\begin{aligned}&\sqrt{\theta}\left(\frac{\gamma-\log(4\pi\theta)}{\theta}-4\sum_{n=1}^\infty d(n)K_0(2\pi n\theta)\right)\\ &\qquad\qquad=\frac{1}{\sqrt{\theta}}\left(\left\{\gamma-\log\frac{4\pi}{\theta}\right\}\theta-4\sum_{n=1}^\infty d(n)K_0\left(\frac{2\pi n}{\theta}\right)\right).\end{aligned}\tag{4.95}$$

This clearly implies that

$$\frac{1}{\theta}\left(\sum_{n=1}^\infty d(n)K_0\left(\frac{2\pi n}{\theta}\right)-\left\{\frac{\gamma-\log\frac{4\pi}{\theta}}{4}\right\}\right)=\sum_{n=1}^\infty d(n)K_0(2\pi n\theta)-\frac{\gamma-\log(4\pi\theta)}{4\theta}.$$

Thus the final result is

$$\frac{8}{\sqrt{\theta}}\int_0^\infty\frac{\Xi^2(\frac{t}{2})}{(1+t^2)^2}\cos\left(\frac{1}{2}t\log\theta\right)dt=\pi\left(\sum_{n=1}^\infty d(n)K_0(2\pi n\theta)-\frac{\gamma-\log(4\pi\theta)}{4\theta}\right).$$

After simplifications, the corollary follows.

4.8.3 Proof of Corollary 4.21

The function

$$K_{/\nu}(z):=\frac{1}{\pi}\sum_{m=0}^\infty\frac{(z/2)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)}\left\{2\log\left(\frac{1}{2}z\right)-\frac{\Gamma'}{\Gamma}(m+1)-\frac{\Gamma'}{\Gamma}(m+\nu+1)\right\}.$$

was studied extensively by Dixon and Ferrar in [DF34, DF35]. In [DF35], they proved that under the Koshliakov kernel introduced earlier one has

$$\int_0^\infty\left(\frac{2}{\pi}K_0(a\sqrt{t})-Y_0(a\sqrt{t})\right)\exp(-bt)dt=-\frac{\sqrt{\pi}}{b}K_{/-\frac{1}{2}}\left(\frac{a^2}{4b}\right),$$

as well as

$$-\sqrt{\pi}\int_0^\infty\left(\frac{2}{\pi}K_0(a\sqrt{t})-Y_0(a\sqrt{t})\right)K_{/-\frac{1}{2}}(bt)dt=\frac{1}{b}\exp\left(-\frac{a^2}{4b}\right).$$

They also link this function to the logarithmic integral by proving

$$K_{/-\frac{1}{2}}(x) = \mathcal{L}(x).$$

Therefore, for $\theta > 0$ we may choose the pair of functions

$$\varphi(x) = \exp(-\theta x) \quad \text{and} \quad \psi(x) = -\frac{2\sqrt{\pi}}{\theta} K_{/-\frac{1}{2}}\left(\frac{4x}{\theta}\right),$$

and these will satisfy (4.71) and (4.72). To compute the Mellin transforms one first needs to see that $\mathrm{li}(e^x) = \mathrm{Ei}(x)$, where

$$\mathrm{Ei}(x) := -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt,$$

is the exponential integral function. The Mellin transform of $e^x \,\mathrm{Ei}(-x)$ is given by entry 13.10 of [Obe74]

$$\int_0^\infty e^x \,\mathrm{Ei}(-x) x^{s-1} dx = -\pi \csc(\pi s)\Gamma(s), \quad 0 < \mathrm{Re}(s) < 1.$$

Thus, from (4.79) we obtain

$$\int_0^\infty K_{/-\frac{1}{2}}(x) x^{s-1} dx = -\pi^{-1/2} \cot\left(\frac{\pi s}{2}\right)\Gamma(s).$$

valid for $0 < \mathrm{Re}(s) < 1$. Finally, it is now clear that

$$Z_1(s) = \frac{\theta^{-s}}{\Gamma^2(\frac{s}{2})}\Gamma(s) \quad \text{and} \quad Z_2(s) = \frac{2^{1-2s}\theta^{s-1}}{\Gamma^2(\frac{s}{2})} \cot\left(\frac{\pi s}{2}\right)\Gamma(s), \tag{4.96}$$

for $\mathrm{Re}(s) > 0$ and $0 < \mathrm{Re}(s) < 1$, respectively. The last expression for $Z_2(s)$ can be analytically continued to $\mathrm{Re}(s) < 2$. Putting these two together yields

$$\mathcal{F}_\theta(t) = Z\left(\frac{1+it}{2}\right).$$

Thus, we end up with Corollary 4.21.

4.9 The Ramanujan case

4.9.1 Proof of Theorem 4.20

Let us consider the usual integral

$$\mathcal{R}(Z) = \int_0^\infty f\left(\frac{t}{2}\right) \Xi^2\left(\frac{t}{2}\right) Z\left(\frac{1+it}{2}\right) dt,$$

where again $f(t) = g(it)g(-it)$ with g an analytic function of t. Setting $s = (1+it)/2$ yields

$$\mathcal{R}(Z) = \frac{1}{i}\int_{(\frac{1}{2})} g\left(s - \frac{1}{2}\right) g\left(\frac{1}{2} - s\right) \xi^2(s) Z(s) ds.$$

Now choose

$$g(s) = \Gamma\left(-\frac{1}{4} + \frac{s}{2}\right)\left(s + \frac{1}{2}\right)^{-1} \quad \text{so that} \quad g\left(s - \frac{1}{2}\right) g\left(\frac{1}{2} - s\right) = \frac{\Gamma(\frac{s-1}{2})\Gamma(-\frac{s}{2})}{s(1-s)}.$$

This in turn implies that

$$\begin{aligned}\mathcal{R}(Z) &= 4\int_0^\infty \left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \Xi^2\left(\frac{t}{2}\right) Z\left(\frac{1+it}{2}\right)\frac{dt}{1+t^2} \\ &= -\frac{1}{i}\int_{(\frac{1}{2})} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\frac{s(s-1)}{4}\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)Z(s)ds.\end{aligned}$$

We deform the path of integration past the region of absolute convergence of the Riemann zeta-function at $s = 1 + \varepsilon$ with $\varepsilon > 0$. In doing so, we pick up a double pole at $s = 1$ so that

$$\begin{aligned}\mathcal{R}(Z) &= -\frac{1}{i}\int_{(1+\varepsilon)} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\frac{s(s-1)}{4}\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)Z(s)ds \\ &+ 2\pi \operatorname*{res}_{s=1} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\frac{s(s-1)}{4}\pi^{-s}\Gamma^2\left(\frac{s}{2}\right)\zeta^2(s)Z(s) \\ &= -2\pi\sum_{n=1}^\infty d(n)\frac{1}{2\pi i}\int_{(1+\varepsilon)} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\frac{s(s-1)}{4}(\pi n)^{-s}\Gamma^2\left(\frac{s}{2}\right)Z(s)ds \\ &- 2\pi^{3/2}((\gamma - \log 2\pi)Z(1) + Z'(1)).\end{aligned}$$

Let us now use Parseval's form (A.28) with $w = \pi n$ and

$$F(s) = \Gamma^2\left(\frac{s}{2}\right)Z(s),$$

so that

$$f(x) = \frac{1}{2\pi i}\int_{(\sigma)} F(s)x^{-s}ds = \frac{1}{2\pi i}\int_{(\sigma)} \Gamma^2\left(\frac{s}{2}\right)Z(s)x^{-s}ds = \Theta(x),$$

with $\sigma > 0$ as well as

$$G(s) = \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\frac{s(s-1)}{4}.$$

To compute $g(x)$ we note that

$$\frac{1}{2\pi i}\int_{(k)} \Gamma(s)\Gamma(z-s)x^{-s}ds = \frac{\Gamma(z)}{(1+x)^z},$$

for $0 < k = \mathrm{Re}(s) < \mathrm{Re}(z)$, which we can use to see that

$$\begin{aligned}g(x) &= \frac{1}{2\pi i}\int_{(\sigma)} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\frac{s(s-1)}{4}x^{-s}ds \\ &= -\frac{1}{2\pi i}\int_{(\sigma)} \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{2}\right)x^{-s}ds = -\frac{x\sqrt{\pi}}{(1+x^2)^{3/2}}.\end{aligned}$$

Therefore this means that

$$\begin{aligned}\mathcal{R}(Z) &= -2\pi\sum_{n=1}^{\infty} d(n)\frac{1}{2\pi i}\int_{(1+\varepsilon)} F(s)G(s)(\pi n)^{-s}ds - 2\pi^{3/2}((\gamma-\log 2\pi)Z(1)+Z'(1)) \\ &= 2\pi^{5/2}\sum_{n=1}^{\infty} nd(n)\int_0^{\infty}\frac{\Theta(x)x}{(x^2+\pi^2n^2)^{3/2}}dx - 2\pi^{3/2}((\gamma-\log 2\pi)Z(1)+Z'(1)).\end{aligned}$$

After simplifications, we obtain the result we were seeking. The same mechanism as in (4.94) makes the integrals along the horizontal segments tend to zero as $t\to\infty$.

4.9.2 Proof of Corollary 4.4

Before choosing the pair of φ,ψ, let us notice the following. For $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $\mathrm{Re}(\nu)>-1$ and $\mu\in\mathbb{C}$, one has [GR07, entry 6.565.7]

$$\int_0^{\infty} x^{1+\nu}(x^2+a^2)^{\mu}K_{\nu}(bx)dx = 2^{\nu}\Gamma(\nu+1)a^{\nu+\mu+1}b^{-1-\mu}S_{\mu-\nu,\mu+\nu+1}(ab),$$

where $S_{\mu,\nu}(z)$ is the Lommel function. However, from the footnote on the first page of [Gla10] we also have

$$S_{\mu,\nu}(z) = z^{\mu+1}\int_0^{\infty} ye^{-yz}{}_2F_1\left(\frac{1-\mu+\nu}{2},\frac{1-\mu-\nu}{2},\frac{3}{2};-y^2\right)dy,$$

for all $\mu,\nu\in\mathbb{C}$. Thus, for $\mu=-3/2$ and $\nu=-1/2$ we obtain

$$S_{-3/2,-1/2}(z) = z^{-1/2}\int_0^{\infty} ye^{-yz}{}_2F_1\left(1,\frac{3}{2},\frac{3}{2};-y^2\right)dy.$$

Moreover, by [GR07, entry 9.121.1]

$${}_2F_1\left(1,\frac{3}{2},\frac{3}{2};-y^2\right) = \frac{1}{1+y^2}$$

for all reals y such that $y\neq 1$. Hence

$$\int_0^{\infty}\frac{xK_0(2\theta x)}{(x^2+t^2)^{3/2}}dx = \sqrt{\frac{2\theta}{t}}S_{-3/2,-1/2}(2\theta t) = \frac{1}{t}\int_0^{\infty}\frac{xe^{-2x\theta}}{x^2+t^2}dx, \tag{4.97}$$

for $t>0$ and $\theta>0$. As an aside, we note that by [AS72, p. 232] one has

$$\int_0^{\infty}\frac{xe^{-zx}}{x^2+1}dx = -\operatorname{Ci}(z)\cos(z)+\left(\frac{\pi}{2}-\operatorname{Si}(z)\right)\sin(z).$$

Next, we choose the pair $\varphi(x)=\psi(x)=K_0(2x)$ so that

$$\begin{aligned}&\frac{-1}{\pi^{3/2}\sqrt{\theta}}\int_0^{\infty}\left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2\Xi^2\left(\frac{t}{2}\right)\cos\left(\frac{1}{2}t\log\theta\right)\frac{dt}{1+t^2} \\ &\quad= \frac{\gamma-\log 2\pi}{4}\left(1+\frac{1}{\theta}\right)+\left(1-\frac{1}{\theta}\right)\frac{\log\theta}{4}\end{aligned}$$

$$
\begin{aligned}
&- \pi \sum_{n=1}^{\infty} nd(n) \int_0^{\infty} x \frac{K_0(2\theta x) + \theta^{-1} K_0(2\theta^{-1} x)}{(x^2+\pi^2 n^2)^{3/2}} dx \\
&= \frac{\gamma - \log 2\pi}{4}\left(1+\frac{1}{\theta}\right) + \left(1-\frac{1}{\theta}\right)\frac{\log\theta}{4} - \sum_{n=1}^{\infty} d(n) \int_0^{\infty} \frac{xe^{-2\theta x} + x\theta^{-1}e^{-2\theta^{-1}x}}{x^2+\pi^2n^2} dx \\
&= \frac{\gamma - \log 2\pi}{4}\left(1+\frac{1}{\theta}\right) + \left(1-\frac{1}{\theta}\right)\frac{\log\theta}{4} \\
&- \int_0^{\infty} (xe^{-2\pi\theta x} + x\theta^{-1}e^{-2\pi\theta^{-1}x}) \sum_{n=1}^{\infty} \frac{d(n)}{x^2+n^2} dx,
\end{aligned}
$$

by the use of (4.97) in the third line and where we have used absolute convergence and the fact that we have the terms $e^{-2x\theta}$ and $e^{-2x\theta^{-1}}$ in the fourth line. The next observation comes from [Dix13c, eq. (1.12)] which states that

$$
\int_0^{\infty} xe^{-2\pi\theta x} \sum_{n=1}^{\infty} \frac{d(n)}{x^2+n^2} dx = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}(n\theta) + \frac{1}{2n\theta} - \log(n\theta)\right).
$$

Consequently, when inserting this result we arrive at

$$
\begin{aligned}
&\frac{-1}{\pi^{3/2}\sqrt{\theta}} \int_0^{\infty} \left|\Gamma\left(\frac{-1+it}{4}\right)\right|^2 \Xi^2\left(\frac{t}{2}\right) \cos\left(\frac{1}{2}t\log\theta\right) \frac{dt}{1+t^2} \\
&= \sqrt{\theta}\Bigg[\frac{\gamma - \log\frac{2\pi}{\theta}}{4} + \frac{\gamma - \log 2\pi\theta}{4\theta} + \frac{1}{2}\sum_{m=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}(m\theta) + \frac{1}{2m\theta} - \log(m\theta)\right) \\
&\qquad + \frac{1}{2\theta}\sum_{m=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(\frac{m}{\theta}\right) + \frac{\theta}{2m} - \log\frac{m}{\theta}\right)\Bigg] \\
&= \frac{1}{2}\Bigg[\sqrt{\theta}\Bigg(\frac{\gamma - \log 2\pi\theta}{2\theta} + \sum_{m=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}(m\theta) + \frac{1}{2m\theta} - \log(m\theta)\right)\Bigg) \\
&\qquad + \frac{1}{\sqrt{\theta}}\Bigg(\frac{\gamma - \log\frac{2\pi}{\theta}}{2/\theta} + \sum_{m=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(\frac{m}{\theta}\right) + \frac{\theta}{2m} - \log\frac{m}{\theta}\right)\Bigg)\Bigg].
\end{aligned}
$$

Lastly, we employ the transformation formula given in [BD10, §4], namely

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}(nz) + \frac{1}{2nz} - \log(nz)\right) + \frac{\gamma - \log 2\pi z}{2z} \\
&\qquad = \frac{1}{z}\sum_{n=1}^{\infty}\left(\frac{\Gamma'}{\Gamma}\left(\frac{n}{z}\right) + \frac{z}{2n} - \log\frac{n}{z}\right) + \frac{\gamma - \log\frac{2\pi}{z}}{2}, \quad (4.98)
\end{aligned}
$$

and, after simplifications, this gives the desired result.

Chapter 5

Twisted second moments of the Riemann ζ-function

5.1 Introduction

In [BCHB85], Balabsuramanian, Conrey and Heath-Brown computed the twisted second moment of the Riemann zeta-function

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \overline{\psi}\psi(\tfrac{1}{2}+it)dt \tag{5.1}$$

where ψ is a Dirichlet polynomial of the type

$$\psi(s) = \sum_{n \le T^\theta} \frac{a(n)}{n^s}, \tag{5.2}$$

and $a(n) \ll_\varepsilon n^\varepsilon$. The length T^θ of the polynomial is sensitive to the nature of the coefficients $a(n)$. They also obtained an explicit main term in their theorem for a particular choice of $\psi(s)$.

In [Con89], in order to obtain a higher percentage of zeros of the Riemann zeta-function on the critical line, Conrey needed to establish such type of second moment. In his result he made an ingenious choice of $a(n)$ which allowed him to push the value of θ from 1/2 (see [Lev74]) to 4/7. The possibility of obtaining a mollifier by combining two Dirichlet polynomials of different shape had been considered by Lou [Lou79]. In [BCY11], Bui, Conrey, and Young extended (5.1) with an explicit main term for a more sophisticated choice of $a(n)$. They considered $\psi(s)$ as a convex combination of two Dirichlet polynomials of different shape. Introducing such two-piece mollifier increases the complexity and technicality of the computation of the main term. Another such two-piece mollifier was introduced by Feng [Fen12] and the main term was computed explicitly.

Crucial ingredients to obtaining the error term in [Fen12] were Lemmas 1 and 2. To reach $\theta_1 < 4/7 - \varepsilon$ in [Con89], it was required that $a(n) = \mu(n)F(n)$, for a smooth function F. In [Fen12], the coefficient $a(n)$ in the mollifier was not of the form $\mu(n)F(n)$, for some smooth function F, and it is not clear how the techniques of [Con89] can be directly applied to the proofs of Lemmas 1 and 2 of [Fen12].

Independently of each other, in [BCY11] and [Fen12], the possibility of obtaining a $\psi(s)$ by combining these three Dirichlet polynomials of different shape was mentioned. One can obtain the main term of (5.1) for such choice of $\psi(s)$ by going over some subtle technicalities in the calculations.

In the present chapter we introduce a new mollifier $\psi(s)$ which is a convex combination of four Dirichlet polynomials of different shape. Let

$$\chi(s) := \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right).$$

We will use the convention

$$P_i[n] := P_i\left(\frac{\log(y_i/n)}{\log y_i}\right) \quad \text{and} \quad \tilde{P}_k[n] := \tilde{P}_k\left(\frac{\log(y_4/n)}{\log y_4}\right), \tag{5.3}$$

where P's are polynomials. Recall that $\mu(n)$ denotes the Möbius function, also $\mu_2(n)$ and $\mu_3(n)$ will denote the coefficients in the Dirichlet series of $1/\zeta^2(s)$ and $1/\zeta^3(s)$, respectively, for $\mathrm{Re}(s) > 1$. Also, let $d_k(n)$ denote the number of ways an integer n can be written as a product of $k \geq 2$ fixed factors. Note that $d_1(n) = 1$ and that $d_2(n) = d(n)$ is the number of divisors of n. With this in mind, we define

$$\psi(s) := \psi_1(s) + \psi_2(s) + \psi_3(s) + \psi_4(s), \tag{5.4}$$

where

$$\psi_1(s) = \sum_{n\leq y_1}\frac{\mu(n)n^{\sigma_0-1/2}}{n^s}P_1[n] \tag{5.5}$$

introduced in [Con89],

$$\psi_2(s) = \chi(s+\tfrac{1}{2}-\sigma_0)\sum_{hk\leq y_2}\frac{\mu_2(h)h^{\sigma_0-1/2}k^{1/2-\sigma_0}}{h^s k^{1-s}}P_2[hk] \tag{5.6}$$

introduced in [BCY11],

$$\psi_3(s) = \chi^2(s+\tfrac{1}{2}-\sigma_0)\sum_{hk\leq y_3}\frac{\mu_3(h)d(k)h^{\sigma_0-1/2}k^{1/2-\sigma_0}}{h^s k^{1-s}}P_3[hk] \tag{5.7}$$

introduced in the present chapter, and

$$\psi_4(s) = \sum_{n\leq y_4}\frac{\mu(n)n^{\sigma_0-1/2}}{n^s}\sum_{k=2}^{K}\sum_{p_1\dots p_k|n}\frac{\log p_1\dots\log p_k}{\log^k y_4}\widetilde{P}_k[n], \tag{5.8}$$

introduced in [Fen12]. Here K is a positive integer of our choice and $p_1,\dots,p_k$ are distinct primes. Also we need $P_1(0) = 0$, $P_1(1) = 1$, $P_2(0) = P_2'(0) = P_2''(0) = 0$, $P_3(0) = P_3'(0) = \cdots = P_3^{(6)}(0) = 0$, and $\tilde{P}_k(0) = 0$, for $k = 2,\dots,K$. We use the conventions $y_i = T^{\theta_i}$ and $\sigma_0 = 1/2 - R/\log T$.

The reasoning behind introducing the new piece ψ_3 is that it approximates $1/\zeta(s)$ in some region of the complex plane. We now state our main theorem.

Theorem 5.1. *Let* $\alpha, \beta \ll \frac{1}{\log T}$, $\sigma_0 = \frac{1}{2} - \frac{R}{\log T}$ *and* $R \ll 1$. *Then for* $\theta_1 < 4/7 - \varepsilon$, $\theta_2 < 1/2 - \varepsilon$, $\theta_3 < 3/7 - \varepsilon$, *and* $\theta_4 < 3/7 - \varepsilon$ *we have*

$$I(\alpha, \beta) := \int_1^T \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi}\psi(\sigma_0 + it)dt = CT + O_\varepsilon(T(\log T)^{-1+\varepsilon}), \tag{5.9}$$

where C *is an explicit constant that depends on* $\alpha, \beta, Q, P_1, P_2, P_3, R, \theta_1, \theta_2, \theta_3, \theta_4$ *and* $\tilde{P}_k$ *for* $k = 2, 3, \ldots, K$.

In this chapter we will obtain an explicit formula for the constant C and more specifically we show that

$$\begin{aligned} C = c_{11}(\alpha, \beta) + 2c_{14}(\alpha, \beta) + c_{44}(\alpha, \beta) + c_{22}(\alpha, \beta) + c_{33}(\alpha, \beta) \\ + 2c_{12}(\alpha, \beta) + 2c_{23}(\alpha, \beta) + 2c_{24}(\alpha, \beta), \end{aligned}$$

where the values of $c_{ij}(\alpha, \beta)$ are given in the next section. At the end of this chapter we will provide an application and show that optimizing the numerical value of certain derivatives of C with respect to α and β for specific values of α and β, will give an improved result towards the percentage of zeros of the Riemann zeta-function on the critical line.

5.2 Intermediate Results

From now on we will denote $L = \log T$. Suppose that $w(t)$ is a smooth function with the following properties:

(1) $0 \le w(t) \le 1$ for all $t \in \mathbb{R}$,

(2) $w(t)$ has compact support in $[T/4, 2T]$,

(3) $w^{(j)}(t) \ll_j \Delta^{-j}$ for each $j = 0, 1, 2, \cdots$, where $\Delta = \frac{T}{5L}$.

The Fourier transform of $w(t)$ is denoted by $\widehat{w}(s)$. For $j, k \in \{1, 2, 3, 4\}$ and $(j, k) \notin \{(1, 1), (1, 4), (4, 1), (4, 4)\}$ we define

$$I_{jk}(\alpha, \beta) = \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_j}\psi_k(\sigma_0 + it)dt. \tag{5.10}$$

For $(j, k) \in \{(1, 1), (1, 4), (4, 1), (4, 4)\}$ we define

$$I_{jk}(\alpha, \beta, w) = \frac{1}{\sqrt{\pi}\Delta} \int_{-\infty}^{\infty} e^{-(t-w)^2\Delta^{-2}}\zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_j}\psi_k(\sigma_0 + it)dt. \tag{5.11}$$

The following two propositions were proved in [BCY11, Theorem 3.2 and Theorem 3.3].

Proposition 5.2. *Let* $\theta_1 < 4/7 - \varepsilon$ *and* $\theta_2 < 1/2 - \varepsilon$. *One has that*

$$I_{12}(\alpha, \beta) = c_{12}(\alpha, \beta)\widehat{w}(0) + O(TL^{-1}), \tag{5.12}$$

uniformly for $\alpha, \beta \ll L^{-1}$. *Here* $c_{12}(\alpha,\beta)$ *is given in the main term of* [BCY11, Theorem 3.2].

Proposition 5.3. *Let* $\theta_2 < 1/2 - \varepsilon$. *One has that*

$$I_{22}(\alpha,\beta) = c_2(\alpha,\beta)\widehat{w}(0) + O(TL^{-1+\varepsilon}), \tag{5.13}$$

where $c_2(\alpha,\beta)$ *is given in the main term of* [BCY11, Theorem 3.3].

We will prove the following propositions as intermediate results.

Proposition 5.4. *Let* $\theta_2 < 1/2 - \varepsilon$ *and* $\theta_3 < 1/2 - \varepsilon$. *Then we have*

$$I_{23}(\alpha,\beta) = c_{23}(\alpha,\beta)\widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, *where*

$$\begin{aligned} c_{23}(\alpha,\beta) = \frac{2^8}{7!}\left(\frac{\theta_3}{\theta_2}\right)^6 \frac{d^4}{dx^2dy^2}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 u^4(1-u)^7\left(y_2^{-x}y_3^{au}\right)^{-\alpha}\left(y_2^{y}y_3^{-ub}T\right)^{-\beta} \\ \times P_2''\left(x+y+1-(1-u)\frac{\theta_3}{\theta_2}\right) abP_3^{(6)}((1-a-b)u)dudadb\Bigg]_{x=y=0}. \end{aligned}$$

Also $I_{32}(\alpha,\beta)$ is asymptotic to $I_{23}(\alpha,\beta)$.

Proposition 5.5. *Let* $\theta_3 < 1/2 - \varepsilon$. *Then we have*

$$I_{33}(\alpha,\beta) = c_{33}(\alpha,\beta)\widehat{w}(0) + O(TL^{-1+\varepsilon}),$$

uniformly for $\alpha, \beta \ll L^{-1}$, *where*

$$\begin{aligned} I_{33}(\alpha,\beta) = \frac{2^{12}}{12!}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1\int_0^1 (1-r)^{12} y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))} \\ \times\left(\frac{1}{\theta_3} + (x+y-v(y+r)-u(x+r))\right)(x+r)^2(y+r)^2 \\ \times P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))(Ty_3^{x+y-v(y+r)-u(x+r)})^{-t(\alpha+\beta)} \\ \times dtdrdudv\Bigg)\Bigg|_{x=y=0}. \end{aligned}$$

Proposition 5.6. *Let* K *be an integer greater or equal to* 2 , $\theta_2 < 1/2 - \varepsilon$ *and* $\theta_4 < 1/2 - \varepsilon$. *Then we have*

$$I_{42}(\alpha,\beta) = c_{42}(\alpha,\beta,K)\widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, *where*

$$\begin{aligned} c_{42}(\alpha,\beta,K) = \sum_{k=2}^{K}(c_{42}^{(0,0)}(\alpha,\beta) + c_{42}^{(1,0)}(\alpha,\beta) + c_{42}^{(0,1)}(\alpha,\beta) + c_{42}^{(1,1)}(\alpha,\beta) \\ + c_{42}^{(1,\ge 2)}(\alpha,\beta) + c_{42}^{(\ge 2,1)}(\alpha,\beta) + c_{42}^{(\ge 2,0)}(\alpha,\beta) + c_{42}^{(0,\ge 2)}(\alpha,\beta) + c_{42}^{(\ge 2,\ge 2)}(\alpha,\beta)). \end{aligned}$$

Here we have

$$c_{42}^{(0,0)}(\alpha,\beta) = 4\frac{2^k}{(k+1)!}\frac{d^2}{dxdy}\Big[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^{1+k}\left(y_2^{a-x}y_4^{a(u-1)}\right)^{-\alpha}$$
$$\times \left(y_2^{y-b}y_4^{b(-u+1)}T\right)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times \widetilde{P}_k(x+y+u)P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)dudadb\Big]_{x=y=0},$$

$$c_{42}^{(1,0)}(\alpha,\beta) = -4\frac{2^{k-1}}{(k-1)!}\frac{d}{dy}\Big[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^{k}\left(y_4^{-(1-u)a}y_2^{a}\right)^{-\alpha}$$
$$\times \left(y_4^{b(1-u)+y}y_2^{-b}T\right)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(y+u)dudadb\Big]_{y=0},$$

$$c_{42}^{(0,1)}(\alpha,\beta) = -4\frac{2^{k-1}}{(k-1)!}\frac{d}{dx}\Big[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^{k}\left(y_4^{-(1-u)a-x}y_2^{a}\right)^{-\alpha}$$
$$\times \left(y_4^{b(1-u)}y_2^{-b}T\right)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+u)dudadb\Big]_{x=0},$$

$$c_{42}^{(1,1)}(\alpha,\beta) = 4\frac{2^{k-2}k}{(k-2)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{k-1}$$
$$\times (y_4^{-(1-u)a}y_2^{a})^{-\alpha}(y_4^{b(1-u)}y_2^{-b}T)^{-\beta}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(u)dudadb,$$

$$c_{42}^{(1,\ge 2)}(\alpha,\beta) = -4k!\sum_{l_1+1+l_3=k}\frac{2^{l_1}(-1)^{l_3-2}}{l_1!l_3!(1+l_1)!(l_3-2)!}$$
$$\times \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1 \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times (1-u)^{1+l_1}\left(y_4^{-a(1-u)}y_2^{a}\right)^{-\alpha}\left(y_4^{b(1-u)-uc}y_2^{-b}T\right)^{-\beta}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k((1-c)u)u^{l_3-1}c^{l_3-2}dudcdadb,$$

with $l_3\ge 2$,

$$c_{42}^{(\ge 2,1)}(\alpha,\beta) = -4k!\sum_{l_1+l_2+1=k}\frac{2^{l_1}(-1)^{l_2-2}}{l_1!l_2!(1+l_1)!(l_2-2)!}$$

$$\times \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 \left(1-(1-u)\frac{\theta_4}{\theta_4}\right)^2 (1-u)^{1+l_1}$$
$$\times (y_4^{-a(1-u)+uc} y_2^a)^{-\alpha} (y_4^{b(1-u)} y_2^{-b} T)^{-\beta}$$
$$\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_4}\right)\right)\widetilde{P}_k((1-c)u)u^{l_2-1}c^{l_2-2}dudcdadb,$$

with $l_2 \geq 2$,

$$c_{42}^{(\geq 2,0)}(\alpha,\beta) = 4k! \sum_{l_1+l_2=k} \frac{2^{l_1}(-1)^{l_2}}{l_1!l_2!(l_2-2)!(1+l_1)!}\frac{d}{dx}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 (1-u)^{1+l_1}$$
$$\times (x+u)^{l_2-1}(y_4^{c(u+x)-(1-u)a} y_2^a)^{-\alpha}(y_4^{x+(1-u)b} y_2^{-b}T)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times c^{l_2-2}P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(x+u))dadbdcdu\Bigg]_{x=0}$$

with $l_2 \geq 2$,

$$c_{42}^{(0,\geq 2)}(\alpha,\beta) = 4k! \sum_{l_1+l_3=k} \frac{2^{l_1}(-1)^{l_3}}{l_1!l_3!(l_3-2)!(1+l_1)!}\frac{d}{dy}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 \int_0^1 (1-u)^{1+l_1}$$
$$\times (y+u)^{l_3-1}(y_4^{-y-a(1-u)} y_2^a)^{-\alpha}(y_4^{-c(u+y)+b(1-u)} y_2^{-b}T)^{-\beta}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times c^{l_3-2}P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(y+u))dadbdcdu\Bigg]_{y=0}$$

with $l_3 \geq 2$, *and*

$$c_{42}^{(l_2,l_3)}(\alpha,\beta) = 4k! \sum_{l_1+l_2+l_3=k} \frac{2^{l_1}(-1)^{l_2+l_3}}{l_1!l_2!l_3!(1+l_1)!(l_2-2)!(l_3-2)!}$$
$$\times \iiiint_{\substack{0\le a+b\le 1\\ 0\le g+h\le 1\\ a,b,g,h\ge 0}} \int_0^1 (1-u)^{k+l-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$
$$\times (y_4^{au+gu-a} y_2^a)^{-\alpha}(y_4^{-bu-hu+b} y_2^{-b}T)^{-\beta} P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)$$
$$\times \widetilde{P}_k((1-g-h)u)u^{l_2+l_3-2}g^{l_2-2}h^{l_3-2}dudadbdgdh,$$

with $l_2 \geq 2$ *and* $l_3 \geq 2$.

Also note that $I_{24}(\alpha,\beta)$ is asymptotic to $I_{42}(\alpha,\beta)$.

Proposition 5.7. *Let* $\theta_1 < 4/7-\varepsilon$, $\theta_4 < 3/7-\varepsilon$ *and* $T/2 \leq w \leq T$. *One has that*

$$I_{11}(\alpha,\beta,w) + 2I_{14}(\alpha,\beta,w) + I_{44}(\alpha,\beta,w) = c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta) + O_\varepsilon(L^{-1+\varepsilon}),$$

uniformly for $\alpha,\beta \ll L^{-1}$. *Here* $c_{11}(\alpha,\beta)+2c_{14}(\alpha,\beta)+c_{44}(\alpha,\beta)$ *is given in the main term of* [Fen12, Eq. (5.1)]. *Note that the right-hand side is independent of* w.

Proposition 5.8. *Let* $\theta_1 < 4/7 - \varepsilon$ *and* $\theta_3 < 3/7 - \varepsilon$. *One has that*

$$I_{13}(\alpha, \beta) = O(TL^{-1+\varepsilon}), \tag{5.14}$$

uniformly for $\alpha, \beta \ll L^{-1}$.

Proposition 5.9. *Let* $\theta_1 < 4/7 - \varepsilon$ *and* $\theta_4 < 3/7 - \varepsilon$. *One has that*

$$I_{34}(\alpha, \beta) = O(TL^{-1+\varepsilon}), \tag{5.15}$$

uniformly for $\alpha, \beta \ll L^{-1}$.

Now we choose a $w(t)$ that satisfies (1)-(3), an upper bound (or lower bound) for the characteristic function in the interval $[T/2, T]$, and with support in $[T/2 - \Delta, T + \Delta]$. We note that in this case $\widehat{w}(0) = T/2 + O(T/L)$. Therefore one can see that

$$\int_{T/2}^{T} \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_j}\psi_k(\sigma_0 + it)dt \tag{5.16}$$

can be bounded by

$$\int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi}_j\psi_k(\sigma_0 + it)dt$$

for above choice of w and $(j, k) \notin \{(1, 1), (1, 4), (4, 1), (4, 4)\}$. Using Propositions 5.4, 5.5, and 5.6 we can see that (5.16) can be bounded by $c_{jk}(\alpha, \beta)T/2 + O(T/L)$. Now summing over dyadic segments gives the required asymptotic for (5.16) with the limits of integration replaced by 1 to T. Let $T/4 \leq T_1 < T_2 < 2T$ and we define

$$w(t, T_1, T_2) = \frac{1}{\sqrt{\pi}\Delta} \int_{T_1}^{T_2} e^{-(t-w)^2\Delta^{-2}} dt. \tag{5.17}$$

Then clearly

(a) $0 \leq w(t, T_1, T_2) \leq 1$

(b) $w(t, T_1, T_2) = O(\exp(-\log^2 T))$ when $t \notin [T_1 - \Delta \log T, T_2 + \Delta \log T]$

(c) $w(t, T_1, T_2) = 1 + O(\exp(-\log^2 T))$ when $t \in [T_1 - \Delta \log T, T_2 + \Delta \log T]$.

Now we can select two such $w(t, T_1, T_2)$'s, specifically $w(t, T/2 - \Delta \log T, T + \Delta \log T)$ and $w(t, T/2 + \Delta \log T, T - \Delta \log T)$. Then from the above facts, Proposition 5.7, and (5.17) we bound

$$\sum_{(j,k)\in\{(1,1),(1,4),(4,1),(4,4)\}} \int_{T/2}^{T} \zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_j}\psi_k(\sigma_0 + it)dt \tag{5.18}$$

by $(c_{11}(\alpha, \beta) + 2c_{14}(\alpha, \beta) + c_{44}(\alpha, \beta))T/2 + O_\varepsilon(TL^{-1+\varepsilon})$. Now summing over dyadic segments gives the required asymptotic for (5.17) with the limits of integration replaced by 1 to T.

Since $I(\alpha,\beta)$ is the sum of the terms of the form given in (5.16) and (5.17) with the limits of integration replaced by 1 to T, the equality

$$\begin{aligned} C = c_{11}(\alpha,\beta) + 2c_{14}(\alpha,\beta) + c_{44}(\alpha,\beta) + c_{22}(\alpha,\beta) + c_{33}(\alpha,\beta) \\ + 2c_{12}(\alpha,\beta) + 2c_{23}(\alpha,\beta) + 2c_{24}(\alpha,\beta), \end{aligned}$$

holds.

5.3 Auxiliary lemmas

In this section we collect all the tools, new and old, that will be needed for the forthcoming computations. Throughout this chapter, the notation $\int_{(c)}$ will signify $\int_{c-i\infty}^{c+i\infty}$. The following results were proved in [BCY11].

Lemma 5.10. *Let* $\sigma_{\alpha,-\beta}(l) = \sum_{ab=l} a^{-\alpha}b^{\beta}$. *For* $L^2 \le |t| \le 2T$ *and uniformly for* $\alpha,\beta \ll L^{-1}$,

$$\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it) = \sum_{l=1}^{\infty} \frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}} e^{-l/T^3} + O(T^{-1+\varepsilon}).$$

Lemma 5.11. *Suppose* $w(t)$ *satisfies* (1)-(3), *and* a *and* b *are positive integers with* $ab \le T^{1-\varepsilon}$. *Then, uniformly for* $\alpha,\beta \ll L^{-1}$, *we have*

$$\int_{-\infty}^{\infty} \left(\frac{a}{b}\right)^{-it} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)dt = \sum_{am=bn} \frac{1}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} V_t(mn)w(t)dt \tag{5.19}$$

$$+ \sum_{am=bn} \frac{1}{m^{1/2-\beta}n^{1/2-\alpha}} \int_{-\infty}^{\infty} V_t(mn)\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} w(t)dt + O(T^{-1/2}).$$

Here $V_t(x)$ *is given by*

$$V_t(x) = \frac{1}{2\pi i}\int_{(1)} \left(\frac{t}{2\pi x}\right)^{z} \frac{G(z)}{z}dz,$$

where

$$G(z) = e^{z^2}p(z) \quad \text{and} \quad p(z) = \frac{(\alpha+\beta)^2-(2z)^2}{(\alpha+\beta)^2}.$$

Lemma 5.12. *Suppose that* $z \le x$, $|s| \le \frac{1}{\log x}$, k *is a positive integer, and let* F *and* H *be smooth in an interval containing* $[0,1]$. *Then*

$$\begin{aligned} &\sum_{n\le z} \frac{d_k(n)}{n^{1+s}} F\left(\frac{\log x/n}{\log x}\right) H\left(\frac{\log z/n}{\log z}\right) \\ &= \frac{(\log z)^k}{(k-1)!z^s}\int_0^1 (1-u)^{k-1}F\left(1-(1-u)\frac{\log z}{\log x}\right)H(u)z^{us}du + O((\log 3z)^{k-1}). \end{aligned}$$

Lemma 5.13. *Suppose that* $-1 \le \sigma \le 0$. *Then*

$$\sum_{n\le x} \frac{d_k(n)}{n}\left(\frac{x}{n}\right)^{\sigma} \ll_k (\log 3x)^{k-1}\min(|\sigma|^{-1}, \log 3x).$$

As an extension to the above lemma and following argument to that of [BCY11, Lemma 4.6] we have:

Lemma 5.14. *Suppose that* $-1 \leq \sigma \leq 0$. *Then*

$$\sum_{n \leq x} \frac{(d_k * \Lambda * \cdots * \Lambda)(n)}{n} \left(\frac{x}{n}\right)^{\sigma} \ll_{k,l} (\log 3x)^{k+l-1} \min(|\sigma|^{-1}, \log 3x),$$

where the convolution of Λ *is taken* l *times.*

We also need the following lemma which is an extension of Lemma 5.12.

Lemma 5.15. *Under the conditions of Lemma 5.12, one has*

$$\begin{aligned} S_{k,l} &= \sum_{n \leq z} \frac{(d_k * \Lambda * \cdots * \Lambda)(n)}{n^{1+s}} F\left(\frac{\log x/n}{\log x}\right) H\left(\frac{\log z/n}{\log z}\right) \\ &= \frac{(\log z)^{k+l}}{(k+l-1)! z^s} \int_0^1 (1-u)^{k+l-1} F\left(1 - (1-u)\frac{\log z}{\log x}\right) H(u) z^{us} du \\ &\quad + O((\log 3z)^{k+l-1}), \end{aligned}$$

where the convolution of Λ *is taken* l *times.*

Proof. For $l = 1$ we have

$$\begin{aligned} S_{k,1} &= \sum_{n \leq z} \frac{(d_k * \Lambda)(n)}{n^{1+s}} F\left(\frac{\log x/n}{\log x}\right) H\left(\frac{\log z/n}{\log z}\right) \\ &= \sum_{m \leq z} \frac{\Lambda(m)}{m^{1+s}} \sum_{q \leq z/m} \frac{d_k(q)}{q^{1+s}} F\left(\frac{\log \frac{x}{qm}}{\log x}\right) H\left(\frac{\log \frac{z}{qm}}{\log z}\right) \end{aligned}$$

By Lemma 5.12, we then have

$$\begin{aligned} S_{k,1} = &\sum_{m \leq z} \frac{\Lambda(m)}{m^{1+s}} \Bigg[\frac{(\log \frac{z}{m})^k}{(k-1)! (\frac{z}{m})^s} \\ &\quad \times \int_0^1 (1-u)^{k-1} \left(\frac{z}{m}\right)^{us} F\left(\frac{\log \frac{x}{m}}{\log x} (1 - (1-u)) \frac{\log \frac{z}{m}}{\log \frac{x}{m}}\right) H\left(u \frac{\log \frac{z}{m}}{\log z}\right) du \Bigg] \\ &\quad + O((\log 3z)^{k-1}) \\ = &\sum_{m \leq z} \frac{\Lambda(m)}{m^{1+s}} \Bigg[\frac{(\log \frac{z}{m})^k}{(k-1)! (\frac{z}{m})^s} \\ &\quad \times \int_0^1 (1-u)^{k-1} \left(\frac{z}{m}\right)^{us} F\left((1-u)\left(1 - \frac{\log z}{\log x}\right) + u \frac{\log \frac{x}{m}}{\log x}\right) H\left(u \frac{\log \frac{z}{m}}{\log z}\right) du \Bigg] \\ &\quad + O((\log 3z)^{k-1}) \\ = &\frac{(\log z)^k}{(k-1)! z^s} \int_0^1 (1-u)^{k-1} z^{us} \sum_{m \leq z} \frac{\Lambda(m)}{m^{1+us}} \\ &\quad \times F\left((1-u)\left(1 - \frac{\log z}{\log x}\right) + u \frac{\log \frac{x}{m}}{\log x}\right) H\left(u \frac{\log \frac{z}{m}}{\log z}\right) \left(\frac{\log \frac{z}{m}}{\log z}\right)^k du \\ &\quad + O((\log 3z)^{k-1}) \end{aligned}$$

$$= \frac{(\log z)^k}{(k-1)!z^s} \int_0^1 (1-u)^{k-1} \log z \int_0^1 F\Big((1-u)\Big(1-\frac{\log z}{\log x}\Big) + u\Big(1-(1-b)\frac{\log z}{\log x}\Big)\Big)$$
$$\times H(ub) b^k z^{ubs} db du + O((\log 3z)^{k-1}).$$

Hence, we have

$$S_{k,1} = \frac{(\log z)^k}{(k-1)!z^s} \int_0^1 \int_0^1 b^k (1-u)^{k-1} F\left(1-(1-ub)\frac{\log z}{\log x}\right) H(ub) z^{ubs} db du$$
$$+ O((\log 3z)^{k-1}). \tag{5.20}$$

We perform three changes of variables. First, $u = 1 - v$ so that

$$S_{k,1} = \frac{(\log z)^k}{(k-1)!z^s} \int_0^1 \int_0^1 b^k v^{k-1} F\left(1-(1-b(1-v))\frac{\log z}{\log x}\right) H(b(1-v)) z^{b(1-v)s} db dv$$
$$+ O((\log 3z)^{k-1}). \tag{5.21}$$

Second, we set $v = \frac{a}{b}$ so that

$$S_{k,1} = \frac{(\log z)^k}{(k-1)!z^s} \int_0^1 \int_0^b a^{k-1} F\left(1-(1-b(1-\tfrac{a}{b}))\frac{\log z}{\log x}\right) H(b(1-\tfrac{a}{b})) z^{b(1-\frac{a}{b})s} da db$$
$$+ O((\log 3z)^{k-1}). \tag{5.22}$$

Finally, we set $b = u + a$ and we obtain

$$S_{k,1} = \frac{(\log z)^k}{(k-1)!z^s} \iint\limits_{\substack{u+a\le 1\\ a,u\ge 0}} a^{k-1} F\Big(1-(1-u)\frac{\log z}{\log x}\Big) H(u) z^{us} da du + O((\log 3z)^{k-1})$$
$$= \frac{(\log z)^k}{(k-1)!z^s} \int_0^1 (1-u)^{k-1} F\Big(1-(1-u)\frac{\log z}{\log x}\Big) H(u) z^{us} du + O((\log 3z)^{k}).$$

Hence, by induction on l, we obtain

$$S_{k,l} = \frac{(\log z)^{k+1}}{(k+l-1)!z^s} \int_0^1 (1-u)^{k+l-1} F\left(1-(1-u)\frac{\log z}{\log x}\right) H(u) z^{us} du + O((\log 3z)^{k+l-1}),$$

as it was to be shown. □

We shall also need the following Mellin inversion formula. For $n \le y$ one has

$$P[n] = \sum_i \frac{a_i}{(\log y)^i} (\log(y/n))^i = \sum_i \frac{a_i i!}{(\log y)^i} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y}{n}\right)^s \frac{ds}{s^{i+1}}. \tag{5.23}$$

Note that if $n > y$, then the right-hand side vanishes. From the inverse Mellin transform of the gamma function we have

$$e^{-l/T^3} = \frac{1}{2\pi i} \int_{(1)} T^{3z} \Gamma(z) l^{-z} dz. \tag{5.24}$$

5.4 Proof of Proposition 5.4

First we keep in mind that

$$\overline{\psi_2}(\sigma_0+it)=\chi(\tfrac{1}{2}-it)\sum_{hk\leq y_2}\frac{\mu_2(h)}{h^{1/2-it}k^{1/2+it}}P_2[hk],$$

as well as

$$\psi_3(\sigma_0+it)=\chi^2(\tfrac{1}{2}+it)\sum_{mn\leq y_3}\frac{\mu_3(m)d(n)}{m^{1/2+it}n^{1/2-it}}P_3[mn].$$

Inserting this in the integral yields

$$\begin{aligned}I_{23}(\alpha,\beta)=&\int_{-\infty}^{\infty}w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\chi(\tfrac{1}{2}-it)\\&\times\sum_{hk\leq y_2}\frac{\mu_2(h)}{h^{1/2-it}k^{1/2+it}}P_2[hk]\chi^2(\tfrac{1}{2}+it)\sum_{mn\leq y_3}\frac{\mu_3(m)d(n)}{m^{1/2+it}n^{1/2-it}}P_3[mn]dt.\end{aligned}$$

Recalling that $\chi(\frac{1}{2}+it)\chi(\frac{1}{2}-it)=1$, and pulling out the sums we obtain

$$I_{23}(\alpha,\beta)=\sum_{hk\leq y_2}\sum_{mn\leq y_3}\frac{\mu_2(h)\mu_3(m)d(n)}{(hkmn)^{1/2}}P_2[hk]P_3[mn]J_{23}$$

where

$$J_{23}=\int_{-\infty}^{\infty}w(t)\left(\frac{km}{hn}\right)^{-it}\chi(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)dt.$$

We then use the Stirling formula

$$\chi(\tfrac{1}{2}+\beta-it)\chi(\tfrac{1}{2}+it)=\left(\frac{t}{2\pi}\right)^{-\beta}(1+O(t^{-1})),$$

as well as the functional equation $\zeta(\frac{1}{2}+\beta-it)=\chi(\frac{1}{2}+\beta-it)\zeta(\frac{1}{2}-\beta+it)$, which allows us to rewrite J_{23} with the $-\beta$ inside the ζ function, i.e.

$$\begin{aligned}J_{23}&=\int_{-\infty}^{\infty}w(t)\left(\frac{km}{hn}\right)^{-it}\chi(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+\alpha+it)\chi(\tfrac{1}{2}+\beta-it)\zeta(\tfrac{1}{2}-\beta+it)dt\\&=\int_{-\infty}^{\infty}w(t)\left(\frac{km}{hn}\right)^{-it}\left(\frac{t}{2\pi}\right)^{-\beta}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)dt+O(T^{\varepsilon}).\end{aligned}$$

We employ Lemma 5.10 so that

$$\begin{aligned}J_{23}&=\int_{-\infty}^{\infty}w(t)\left(\frac{km}{hn}\right)^{-it}\left(\frac{t}{2\pi}\right)^{-\beta}\left(\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}dt+O(T^{-1+\varepsilon})\right)+O(T^{\varepsilon})\\&=\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2}}e^{-l/T^3}\int_{-\infty}^{\infty}w(t)\left(\frac{kml}{hn}\right)^{-it}\left(\frac{t}{2\pi}\right)^{-\beta}dt+O(T^{\varepsilon}).\end{aligned}$$

Define

$$w_0(t)=w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\quad\text{and}\quad\widehat{w_0}\left(\frac{1}{2\pi}\log\frac{kml}{hn}\right)=\int_{-\infty}^{\infty}w_0(t)\left(\frac{kml}{hn}\right)^{-it}dt.\tag{5.25}$$

Therefore

$$\begin{aligned} I_{23}(\alpha,\beta) = &\sum_{hk\le y_2}\sum_{mn\le y_3}\frac{\mu_2(h)\mu_3(m)d(n)}{(hkmn)^{1/2}}P_2[hk]P_3[mn]\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2}} \\ &\times e^{-l/T^3}\widehat{w_0}\left(\frac{1}{2\pi}\log\frac{kml}{hn}\right) \\ &+ O_\varepsilon(T^\varepsilon(y_2y_3)^{1/2}). \end{aligned}$$

We can bound the off diagonal terms, i.e. those where $kml \neq hn$, in a similar fashion as in the proof of Proposition 5.6.

5.4.1 Main term ($kml = hn$):

From (5.23) and (5.24)

$$\begin{aligned} I_{23}(\alpha,\beta) = \widehat{w_0}(0)&\sum_{i,j}\frac{a_i i! b_j j!}{\log^i y_2 \log^j y_3}\left(\frac{1}{2\pi i}\right)^3\int_{(1)}\int_{(1)}\int_{(1)}T^{3z}\Gamma(z)y_2^s y_3^u \\ &\times\sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{h^{1/2+s}k^{1/2+s}m^{1/2+u}n^{1/2+u}l^{1/2+z}}\frac{dzdsdu}{s^{i+1}u^{j+1}} + O(T^{1-\varepsilon}). \end{aligned}$$

Let

$$S := \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{h^{1/2+s}k^{1/2+s}m^{1/2+u}n^{1/2+u}l^{1/2+z}} \tag{5.26}$$

Since the functions in (5.26) are completely multiplicative, a p-adic analysis shows that

$$S = \frac{\zeta^8(1+s+u)\zeta^2(1+\alpha+u+z)\zeta^2(1-\beta+u+z)}{\zeta^2(1+2s)\zeta^6(1+2u)\zeta^2(1+\alpha+s+z)\zeta^2(1-\beta+s+z)}A(s,u,z). \tag{5.27}$$

A detailed argument for obtaining (5.27) is given in the proof of (5.42). Here $A(s,u,z)$ is a certain arithmetical factor that is given by an Euler product that is absolutely and uniformly convergent in some product of fixed half-planes containing the origin. In particular when $s = u = z$, one has

$$\begin{aligned} A(s,s,s) &= \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{(kmlhn)^{1/2+s}} = \sum_{kml=hn}\frac{\mu_2(h)\mu_3(m)\sigma_{\alpha,-\beta}(l)d(n)}{(kml)^{1+2s}} \\ &= \sum_{j=1}^{\infty}\sum_{kml=j}\frac{\mu_3(m)\sigma_{\alpha,-\beta}(l)}{(kml)^{1+2s}}\sum_{hn|j}\mu_2(h)d(n) = 1, \end{aligned}$$

since $\sum_{hn|j}\mu_2(h)d(n) = 1$ when $j = 1$ and vanishes when $j > 1$. Hence,

$$\begin{aligned} I_{23}(\alpha,\beta) = \widehat{w_0}(0)&\sum_{i,j}\frac{a_i i! b_j j!}{\log^i y_2 \log^j y_3}\left(\frac{1}{2\pi i}\right)^3\int_{(1)}\int_{(1)}\int_{(1)}T^{3z}\Gamma(z)y_2^s y_3^u \\ &\times\frac{\zeta^8(1+s+u)\zeta^2(1+\alpha+u+z)\zeta^2(1-\beta+u+z)}{\zeta^2(1+2s)\zeta^6(1+2u)\zeta^2(1+\alpha+s+z)\zeta^2(1-\beta+s+z)}A(s,u,z)\frac{dzdsdu}{s^{i+1}u^{j+1}} \\ &+ O(T^{1-\varepsilon}). \end{aligned} \tag{5.28}$$

The next step is to deform the s- and u- contours to $\mathrm{Re}(s) = \mathrm{Re}(u) = \delta$, and then deform the z-contour to $-2\delta/3$, where $\delta > 0$ is some fixed constant such that the arithmetical factor converges absolutely. This implies that we pick up a pole at $z = 0$ coming from $\Gamma(z)$. The bound for the integral on the new lines of integration is

$$|\widehat{w_0}(0)|\Big(\frac{y_2y_3}{T^2}\Big)^{\delta} \ll T^{1-\varepsilon}.$$

Consequently, we are left with

$$I_{23}(\alpha,\beta) = \widehat{w_0}(0)\sum_{i,j}\frac{a_i i! b_j j!}{\log^i y_2 \log^j y_3}K_{23} + O(T^{1-\varepsilon}), \tag{5.29}$$

where

$$K_{23} = \Big(\frac{1}{2\pi i}\Big)^2\int_{(\delta)}\int_{(\delta)} y_2^s y_3^u \frac{\zeta^8(1+s+u)\zeta^2(1+\alpha+u)\zeta^2(1-\beta+u)}{\zeta^2(1+2s)\zeta^6(1+2u)\zeta^2(1+\alpha+s)\zeta^2(1-\beta+s)} \times A(s,u,0)\frac{dsdu}{s^{i+1}u^{j+1}}.$$

Let K_{23}' be the same integral as K_{23} but with $A(s,u,0)$ replaced by $A(0,0,0)$. Since $A(s,u,0) = 1 + O(|s|) + O(|u|)$, then $K_{23} = K_{23}' + O(L^{i+j-1})$. The variables s and u are coupled together in the term $\zeta^8(1+s+u)$, so let us replace this by its Dirichlet series and reverse the order of summation and integration. Hence, we get

$$K_{23}' = \sum_{n\le\min(y_2,y_3)}\frac{d_8(n)}{n}K_2K_3,$$

where

$$K_2 = \frac{1}{2\pi i}\int_{(\delta)}\Big(\frac{y_2}{n}\Big)^s\frac{1}{\zeta^2(1+2s)\zeta^2(1+\alpha+s)\zeta^2(1-\beta+s)}\frac{ds}{s^{i+1}},$$

and

$$K_3 = \frac{1}{2\pi i}\int_{(\delta)}\Big(\frac{y_3}{n}\Big)^u\frac{\zeta^2(1+\alpha+u)\zeta^2(1-\beta+u)}{\zeta^6(1+2u)}\frac{du}{u^{j+1}}.$$

The truncation of n is at $\min(y_2, y_3) = y_3$ since $\theta_3 < \theta_2$ and this is accomplished by moving the u-integral to the far right. Let us now compute each integral separately.

Lemma 5.16. *Suppose $i \geq 3$ and $j \geq 7$. Then*

$$K_2 = \frac{4}{(i-2)!}\frac{d^4}{dx^2dy^2}e^{\alpha x-\beta y}\Big(x+y+\log\frac{y_2}{n}\Big)^{i-2}\Big|_{x=y=0} + O(L^{i-7}), \tag{5.30}$$

as well as

$$K_3 = \frac{64(\log y_3/n)^{j-2}}{(j-6)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}(1-a-b)^{j-6}ab\Big(\frac{y_3}{n}\Big)^{-a\alpha+b\beta}dadb + O(L^{j-3}). \tag{5.31}$$

Proof. First we examine K_2. An argument along the lines of the prime number theorem indicates that the integral K_2 is captured by the residue at $s = 0$, with an error of size $(\log y_2/n)^{-A}$ for arbitrarily large A. But since $n \leq y_3$ we have that $\log(y_2/n) \geq$

$\log(y_2/y_3) = (\theta_2 - \theta_3)L$ and hence this error is as desired. Using

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_n (-1)^n \gamma_n (s-1)^n, \tag{5.32}$$

where γ_n are the Stieltjes' constants, indicates that

$$K_2 = 4\frac{1}{2\pi i}\oint \Big(\frac{y_2}{n}\Big)^s (\alpha+s)^2(-\beta+s)^2 \frac{ds}{s^{i-1}} + O(L^{i-7}),$$

where the contour is a small circle enclosing 0. Hence

$$\begin{aligned} K_2 &= 4\frac{d^4}{dx^2dy^2} e^{\alpha x - \beta y} \frac{1}{2\pi i}\oint \Big(\frac{y_2}{n}e^{x+y}\Big)^s \frac{ds}{s^{i-1}}\Big|_{x=y=0} + O(L^{i-7}) \\ &= \frac{4}{(i-2)!}\frac{d^4}{dx^2dy^2} e^{\alpha x-\beta y}\Big(x+y+\log\frac{y_2}{n}\Big)^{i-2}\Big|_{x=y=0} + O(L^{i-7}). \end{aligned}$$

Let us now move on to K_3. As we reasoned previously, the prime number theorem shows that we can replace the contour by a small circle around the origin with radius $\asymp L^{-1}$, with error $O(1)$. On this contour and by the use of (5.32) we obtain

$$K_3 = 64\frac{1}{2\pi i}\oint \Big(\frac{y_3}{n}\Big)^u \frac{1}{(\alpha+u)^2(-\beta+u)^2}\frac{du}{u^{j-5}} + O(L^{j-3}).$$

Note the identity

$$\int_{1/q}^1 r^{\alpha+u-1}\log^\tau r dr = \frac{(-1)^\tau \tau!}{(\alpha+u)^{\tau+1}} - \frac{q^{-\alpha-u}}{(\alpha+u)^{\tau+1}}P(u,\alpha,\log q) \tag{5.33}$$

where P is a polynomial in $\log q$ of degree $\tau - 1$. Set $q = y_3/n$. Only the first term of the right-hand side above contributes when we insert this expression into K_3. This is because the contribution from the second term is

$$64q^{-\alpha}\log q\frac{1}{2\pi i}\oint \frac{q^{-\alpha}(1+(u+\alpha))}{(\alpha+u)^2(-\beta+u)^2}du,$$

which vanishes by taking the contour to be arbitrary large. Then K_3 becomes

$$\begin{aligned} K_3 &= 64\int_{1/q}^1\int_{1/(qr)}^1 r^{\alpha-1}t^{-\beta-1}\log r\log t\frac{1}{2\pi i}\oint (qrt)^u\frac{du}{u^{j-5}}dtdr + O(L^{j-3}) \\ &= \frac{64}{(j-6)!}\int_{1/q}^1\int_{1/(qr)}^1 r^{\alpha-1}t^{-\beta-1}(\log r\log t)\Big(\log\frac{y_3}{n}rt\Big)^{j-6}dtdr + O(L^{j-3}). \end{aligned}$$

Finally, make the change of variables $r = q^{-a}$ and $t = q^{-b}$ so that after simplifications, we get

$$K_3 = \frac{64(\log y_3/n)^{j-2}}{(j-6)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} (1-a-b)^{j-6}ab\Big(\frac{y_3}{n}\Big)^{-a\alpha+b\beta}dadb + O(L^{j-3}).$$

This proves both statements of the lemma. $\square$

The sum over i becomes

$$\begin{aligned}\sum_i \frac{a_i i!}{(\log y_2)^i} K_2 &= \frac{4}{(\log y_2)^2}\frac{d^4}{dx^2dy^2} e^{\alpha x-\beta y}\sum_i a_i i(i-1)\left(\frac{x+y}{\log y_2}+\frac{\log y_2/n}{\log y_2}\right)^{i-2}\Bigg|_{x=y=0} \\ &\quad + O(L^{-7}) \\ &= \frac{4}{(\log y_2)^2}\frac{d^4}{dx^2dy^2} e^{\alpha x-\beta y} P_2{''}\left(\frac{x+y}{\log y_2}+\frac{\log(y_2/n)}{\log y_2}\right)\Bigg|_{x=y=0} + O(L^{-7}).\end{aligned}$$

It is more convenient to write this as

$$\sum_i \frac{a_i i!}{(\log y_2)^i} K_2 = \frac{4}{(\log y_2)^6}\frac{d^4}{dx^2dy^2}\left[y_2^{\alpha x-\beta y} P_2{''}\left(x+y+\frac{\log(y_2/n)}{\log y_2}\right)\right]_{x=y=0} + O(L^{-7}).$$

For the sum over j we get

$$\begin{aligned}\sum_j \frac{b_j j!}{(\log y_3)^j} K_3 &= \sum_j \frac{64 b_j j!}{(\log y_3)^j}\frac{(\log y_3/n)^{j-2}}{(j-6)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} (1-a-b)^{j-6} ab \\ &\qquad \times \left(\log \frac{y_3}{n}\right)^{-a\alpha+b\beta} dadb + O(L^{-3}) \\ &= \frac{64(\log y_3/n)^4}{(\log y_3)^6}\sum_j b_j j(j-1)(j-2)(j-3)(j-4)(j-5) \\ &\qquad \times \iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \left((1-a-b)\frac{\log y_3/n}{\log y_3}\right)^{j-6} ab\left(\frac{y_3}{n}\right)^{-a\alpha+b\beta} dadb \\ &\quad + O(L^{-3}) \\ &= \frac{64(\log y_3/n)^4}{(\log y_3)^6}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \left(\frac{y_3}{n}\right)^{-a\alpha+b\beta} ab P_3^{(6)} \\ &\qquad \times \left((1-a-b)\frac{\log y_3/n}{\log y_3}\right) dadb + O(L^{-3}).\end{aligned}$$

Next, we recall that

$$\widehat{w}_0(0) = T^{-\beta}\widehat{w}(0)(1+O(L^{-1})),$$

and therefore

$$\begin{aligned} I_{23}(\alpha,\beta) &= \frac{\widehat{w}(0)}{T^\beta}\sum_{n\le y_3}\frac{d_8(n)}{n}\frac{4}{(\log y_2)^6}\frac{d^4}{dx^2dy^2}\left[y_2^{\alpha x-\beta y} P_2{''}\left(x+y+\frac{\log(y_2/n)}{\log y_2}\right)\right]_{x=y=0} \\ &\quad \times \frac{64(\log y_3/n)^4}{(\log y_3)^6}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \left(\frac{y_3}{n}\right)^{-a\alpha+b\beta} ab P_3^{(6)}\left((1-a-b)\frac{\log y_3/n}{\log y_3}\right) dadb \\ &\quad + O(T/L) \\ &= \frac{256 T^{-\beta}\widehat{w}(0)}{(\log y_2)^6(\log y_3)^2}\frac{d^4}{dx^2dy^2}\Bigg(y_2^{\alpha x-\beta y}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\sum_{n\le y_3}\frac{d_8(n)}{n}\frac{(\log y_3/n)^4}{(\log y_3)^4} \\ &\qquad \times \left(\frac{y_3}{n}\right)^{-a\alpha+b\beta} ab P_2{''}\left(x+y+\frac{\log(y_2/n)}{\log y_2}\right) \\ &\qquad \times P_3^{(6)}\left((1-a-b)\frac{\log(y_3/n)}{\log y_3}\right) dadb\Bigg)\Bigg|_{x=y=0}\end{aligned}$$

$$+ O(T/L).$$

The last step is to apply Lemma 5.12. We choose $k = 8$, $x = y_2$, $z = y_3$, $F(u) = P_2''(x+y+u)$, $H(u) = u^4 P_3^{(6)}((1-a-b)u)$. These substitutions give

$$\begin{aligned}\sum_{n \le y_3} \frac{d_8(n)}{n^{1-a\alpha+b\beta}} \frac{(\log(y_3/n))^4}{(\log y_3)^4} P_2''\left(x + y + \frac{\log(x/n)}{\log x}\right) P_3^{(6)}\left((1-a-b)\frac{\log(y_3/n)}{\log y_3}\right) \\ = \frac{(\log y_3)^8 (y_3)^{a\alpha - b\beta}}{7!} \int_0^1 (1-u)^7 P_2''\left(x + y + 1 - (1-u)\frac{\log y_3}{\log x}\right) \\ \times u^4 P_3^{(6)}((1-a-b)u)(y_3)^{u(-a\alpha+b\beta)} du + O(\log^7 y_3).\end{aligned}$$

Inserting $y_2 = T^{\theta_2}$ and $y_3 = T^{\theta_3}$ we obtain that

$$\begin{aligned}c_{23}(\alpha,\beta) = \frac{2^8}{7!}\left(\frac{\theta_3}{\theta_2}\right)^6 \frac{d^4}{dx^2 dy^2}\Bigg[\iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \int_0^1 u^4 (1-u)^7 (y_2^{-x} y_3^{au})^{-\alpha} (y_2^{y} y_3^{-ub} T)^{-\beta} \\ \times P_2''\left(x + y + 1 - (1-u)\frac{\theta_3}{\theta_2}\right) ab P_3^{(6)}((1-a-b)u) du\, da\, db\Bigg]_{x=y=0}.\end{aligned}$$

which is precisely the term appearing in Proposition 5.4.

5.5 Proof of Proposition 5.5

One has

$$\overline{\psi_3}(\sigma_0 + it) = \chi^2(\tfrac{1}{2} - it) \sum_{h_1 k_1 \le y_3} \frac{\mu_3(h_1) d(k_1)}{h_1^{1/2-it} k_1^{1/2+it}} P_3[h_1 k_1],$$

as well as

$$\psi_3(\sigma_0 + it) = \chi^2(\tfrac{1}{2} + it) \sum_{h_2 k_2 \le y_3} \frac{\mu_3(h_2) d(k_2)}{h_2^{1/2+it} k_2^{1/2-it}} P_3[h_2 k_2].$$

Inserting these in the integral and pulling out the sums, we obtain

$$\begin{aligned}I_{33}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\chi^2(\tfrac{1}{2}-it) \\ &\times \sum_{h_1 k_1 \le y_3} \frac{\mu_3(h_1) d(k_1)}{h_1^{1/2-it} k_1^{1/2+it}} P_3[h_1 k_1] \chi^2(\tfrac{1}{2}+it) \sum_{h_2 k_2 \le y_3} \frac{\mu_3(h_2) d(k_2)}{h_2^{1/2+it} k_2^{1/2-it}} P_3[h_2 k_2] dt \\ &= \sum_{h_1,k_1,h_2,k_2} \frac{\mu_3(h_1)\mu_3(h_2) d(k_1) d(k_2)}{h_1^{1/2} k_1^{1/2} h_2^{1/2} k_2^{1/2}} P_3[h_1 k_1] P_3[h_2 k_2] \\ &\times \int_{-\infty}^{\infty} \left(\frac{k_1 h_2}{h_1 k_2}\right)^{-it} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it) dt.\end{aligned}$$

We now apply Lemma 5.11. Thus $I_{33}(\alpha,\beta) = I_{33}'(\alpha,\beta) + I_{33}''(\alpha,\beta)$, where I_{33}'' can be obtained from I_{33}' by switching α by $-\beta$ and multiplying by

$$\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} = T^{-\alpha-\beta} + O(L^{-1}),$$

for $t \asymp T$. From (5.23) we have

$$I_{33}'(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{b_i b_j i! j!}{(\log y_3)^{i+j}} \sum_{h_1k_2m=k_1h_2n} \frac{\mu_3(h_1)\mu_3(h_2)d(k_1)d(k_2)}{(h_1k_1h_2k_2)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}}$$
$$\times \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} \left(\frac{y_3}{h_1k_1}\right)^s \left(\frac{y_3}{h_2k_2}\right)^u \left(\frac{t}{2\pi mn}\right)^z \frac{G(z)}{z} dz \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt.$$

Let

$$S := \sum_{h_1k_2m=k_1h_2n} \frac{\mu_3(h_1)\mu_3(h_2)d(k_1)d(k_2)}{(h_1k_1)^{1/2+s}(h_2k_2)^{1/2+u} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \tag{5.34}$$

Evaluating this p-adically (for details see the argument of the proof of (5.42)) one gets

$$S = \frac{\zeta^{13}(1+s+u)\zeta^2(1+\beta+u+z)\zeta^2(1+\alpha+s+z)\zeta(1+\alpha+\beta+2z)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s+z)\zeta^3(1+\alpha+u+z)} B(s,u,z).$$

Again $B(s,u,z)$ is an arithmetical factor converging absolutely and uniformly in a product of half-planes containing the origin. As in the proof of Proposition 5.4, one can show that $B(s,s,s) = 1$. This leaves us with

$$I_{33}'(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{b_i b_j i! j!}{(\log y_3)^{i+j}} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)}$$
$$\times \frac{\zeta^{13}(1+s+u)\zeta^2(1+\beta+u+z)\zeta^2(1+\alpha+s+z)\zeta(1+\alpha+\beta+2z)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s+z)\zeta^3(1+\alpha+u+z)}$$
$$\times B(s,u,z) y_3^{s+u} \left(\frac{t}{2\pi}\right)^z \frac{G(z)}{z} dz \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt.$$

As in the previous computation, the next step is to move contours around carefully and wisely. We take the s-, u- and z- contours of integration to $\delta > 0$ small and then deform z to $-\delta + \varepsilon$ crossing the simple pole of $1/z$ at $z = 0$ only. Recall that $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$. The new path of integration gives a contribution of

$$T^{1+\varepsilon} \left(\frac{y_3^2}{T}\right)^\delta \ll T^{1-\varepsilon}.$$

We end up with

$$I_{33}'(\alpha,\beta) = I_{330}'(\alpha,\beta) + O(T^{1-\varepsilon}),$$

where $I_{330}'(\alpha,\beta)$ corresponds to the residue at $z = 0$, i.e.

$$I_{330}'(\alpha,\beta) = \widehat{w}(0)\zeta(1+\alpha+\beta) \sum_{i,j} \frac{b_i b_j i! j!}{(\log y_3)^{i+j}} J_3,$$

where

$$J_3 = \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)} \int_{(\delta)} \frac{\zeta^{13}(1+s+u)\zeta^2(1+\beta+u)\zeta^2(1+\alpha+s)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s)\zeta^3(1+\alpha+u)}$$
$$\times y_3^{s+u} B(s,u,0) \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}}.$$

Since we want to decouple the function where s and u are present, we use Dirichlet series for $\zeta^{(13)}(1+s+u)$ and then reverse order of integration and summation to obtain

$$J_3 = \sum_{m\le y_3} \frac{d_{13}(m)}{m}\left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} B_{\alpha,\beta}(s,u,0)\left(\frac{y_3}{m}\right)^{s+u}$$
$$\times \frac{\zeta^2(1+\alpha+s)\zeta^2(1+\beta+u)}{\zeta^6(1+2u)\zeta^6(1+2s)\zeta^3(1+\beta+s)\zeta^3(1+\alpha+u)}\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}.$$

Let us now take $\delta \asymp L^{-1}$. We can trivially bound the integrals to show that

$$J_{33} = \sum_{n\le y_3}\frac{d_{13}(n)}{n}\left(\frac{1}{2\pi i}\right)^2 L_1L_2 + O(\log^{i+j-2}T) \ll \log^{i+j-1}T.$$

In particular, this means that we can use a Taylor series so that $B(s,u,0) = B(0,0,0) + O(|s|+|u|)$ and this allows us to write $J_3 = {J_3}' + O(L^{i+j-2})$, say. This process decouples the variables s and u so that

$${J_3}' = \sum_{m\le y_3}\frac{d_{13}(m)}{m}L_1L_2, \tag{5.35}$$

where

$$L_1 = \frac{1}{2\pi i}\int_{(\delta)}\left(\frac{y_3}{m}\right)^s \frac{\zeta^2(1+\alpha+s)}{\zeta^6(1+2s)\zeta^3(1+\beta+s)}\frac{ds}{s^{i+1}}, \tag{5.36}$$

and

$$L_2 = \frac{1}{2\pi i}\int_{(\delta)}\left(\frac{y_3}{m}\right)^u \frac{\zeta^2(1+\beta+u)}{\zeta^6(1+2u)\zeta^3(1+\alpha+u)}\frac{du}{u^{j+1}}.$$

We observe that L_2 is the same as L_1 but with i replaced by j and α and β switched. The result we will need is encapsulated below, its proof follows the proof of Lemma 6.1 of [BCY11].

Lemma 5.17. *With L_1 defined as in (5.36) and for some $\nu \asymp (\log\log y_3)^{-1}$ we have*

$$L_1 = 64\frac{1}{2\pi i}\oint\left(\frac{y_3}{m}\right)^s\frac{(\beta+s)^3}{(\alpha+s)^2}\frac{ds}{s^{i-5}} + O(L^{i-8}) + O\left(\left(\frac{y_3}{m}\right)^{-\nu}L^{\varepsilon}\right),$$

where the contour is a circle of radius $\asymp L^{-1}$ around the origin.

Let us now compute this integral. The result appears below.

Lemma 5.18. *For $i \ge 6$ we have*

$$\frac{1}{2\pi i}\oint\left(\frac{y_3}{m}\right)^s\frac{(\beta+s)^3}{(\alpha+s)^2}\frac{ds}{s^{i-5}} = \frac{1}{(i-6)!}\frac{d^3}{dx^3}\left(x+\log\frac{y_3}{m}\right)^{i-4}e^{\beta x}$$
$$\times \int_0^1 c(1-c)^6\left(\frac{y_3}{m}\right)^{-\alpha c}e^{-\alpha c x}dc\Big|_{x=0}. \tag{5.37}$$

Proof. Using simple derivatives one can write

$$I := \frac{1}{2\pi i}\oint\left(\frac{y_3}{m}\right)^s\frac{(\beta+s)^3}{(\alpha+s)^2}\frac{ds}{s^{i-5}} = \frac{d^3}{dx^3}e^{\beta x}\oint\left(e^x\frac{y_3}{m}\right)^s\frac{1}{(\alpha+s)^2}\frac{ds}{s^{i-5}}\Big|_{x=0}.$$

Let us set $q = e^x y_3/m$, so that

$$I = -\frac{d^3}{dx^3} e^{\beta x} \int_{1/q}^{1} r^{\alpha-1} \log r \left(\frac{1}{2\pi i} \oint (rq)^s \frac{ds}{s^{i-5}} \right) dr \bigg|_{x=0}. \tag{5.38}$$

The second term of (5.33) yields an error which vanishes by taking the contour to be arbitrarily large. Then, by Cauchy's integral formula one has

$$\begin{aligned} I &= \frac{d^3}{dx^3} e^{\beta x} \int_{1/q}^{1} r^{\alpha-1} \log r \frac{1}{(i-6)!} (\log rq)^{i-6} dr \bigg|_{x=0} \\ &= \frac{1}{(i-6)!} \frac{d^3}{dx^3} \left(x + \log \frac{y_3}{m} \right)^{i-4} e^{\beta x} \int_0^1 c(1-c)^6 \left(\frac{y_3}{m} \right)^{-\alpha c} e^{-\alpha c x} dc \bigg|_{x=0}, \end{aligned} \tag{5.39}$$

by the change of variable $r = q^{-c}$. $\square$

Applying Lemmas 5.17 and 5.18 to equation (5.35) yields

$$\begin{aligned} J_3' &= \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} L_1 L_2 \\ &= \frac{2^{12}}{(i-6)!(j-6)!} \frac{d^6}{dx^3 dy^3} e^{x\beta + \alpha y} \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} \left(x + \log \frac{y_3}{m} \right)^{i-4} \left(y + \log \frac{y_3}{m} \right)^{j-4} \\ &\times \int_0^1 \int_0^1 u(1-u)^{i-6} v(1-v)^{j-6} e^{-x\alpha u - y\beta v} \left(\frac{y_3}{m} \right)^{-\alpha u - \beta v} dudv \bigg|_{x=y=0} \\ &+ O(L^{i+j-2}), \end{aligned}$$

where we used Lemma 5.13 to obtain the error term. Hence, telescoping all the way back to I_{33}' and using the Dirichlet series for $\zeta(1+\alpha+\beta)$ gives us

$$\begin{aligned} I_{33}'(\alpha, \beta) &= \frac{\hat{w}(0)}{\alpha+\beta} \sum_{i,j} \frac{b_i b_j i! j!}{(\log y_3)^{i+j}} \frac{2^{12}}{(i-6)!(j-6)!} \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} \frac{d^6}{dx^3 dy^3} e^{x\beta + \alpha y} \\ &\times \left(x + \log \frac{y_3}{m} \right)^{i-4} \left(y + \log \frac{y_3}{m} \right)^{j-4} \int_0^1 \int_0^1 u(1-u)^{i-6} v(1-v)^{j-6} \\ &\times e^{-x\alpha u - y\beta v} \left(\frac{y_3}{m} \right)^{-\alpha u - \beta v} dudv \bigg|_{x=y=0} + O(TL^{\varepsilon-1}) \\ &= \frac{2^{12} \hat{w}(0)}{\alpha+\beta} \frac{d^6}{dx^3 dy^3} \Bigg(\int_0^1 \int_0^1 e^{x(\beta - \alpha u) + y(\alpha - \beta v)} \\ &\times \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} \frac{(x + \log \frac{y_3}{m})^2 (y + \log \frac{y_3}{m})^2}{(\log y_3)^{12}} \\ &\times P_3^{(6)} \left((1-u) \frac{x + \log \frac{y_3}{m}}{\log y_3} \right) P_3^{(6)} \left((1-v) \frac{y + \log \frac{y_3}{m}}{\log y_3} \right) \left(\frac{y_3}{m} \right)^{-\alpha u - \beta v} dudv \Bigg) \bigg|_{x=y=0} \\ &+ O(TL^{\varepsilon-1}) \end{aligned}$$

A more convenient way to write this is as:

$$I_{33}'(\alpha, \beta) = \frac{2^{12} \hat{w}(0)}{(\alpha+\beta) \log^{14} y_3} \frac{d^6}{dx^3 dy^3} \Bigg(\int_0^1 \int_0^1 y_3^{x(\beta - \alpha u) + y(\alpha - \beta v))} \sum_{m \leqslant y_3} \frac{d_{13}(m)}{m} \left(\frac{y_3}{m} \right)^{-\alpha u - \beta v}$$

$$\times \left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)^2\left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)^2$$
$$\times P_3^{(6)}\left((1-u)\left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right)P_3^{(6)}\left((1-v)\left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right)dudv\Bigg)\Bigg|_{x=y=0}$$
$$+O(TL^{\varepsilon-1}).$$

Using Lemma 5.13 with $k=13$, $s=-\alpha u-\beta v$, $x=z=y_3$, $F(r)=(x+r)^2P_3^{(6)}((1-u)(x+r))$ as well as $H(r)=(y+r)^2P_3^{(6)}((1-v)(y+r))$, we then obtain

$$\sum_{m\leqslant y_3}\frac{d_{13}(m)}{m^{1-\alpha u-\beta v}}\left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)^2$$
$$\times P_3^{(6)}\left((1-u)\left(x+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right)\left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)^2P_3^{(6)}\left((1-v)\left(y+\frac{\log\frac{y_3}{m}}{\log y_3}\right)\right)$$
$$=\frac{(\log y_3)^{13}}{12!y_3^{-\alpha u-\beta v}}\int_0^1(1-r)^{12}(x+r)^2P_3^{(6)}((1-u)(x+r))(y+r)^2P_3^{(6)}((1-v)(y+r))$$
$$\times z^{r(-\alpha u-\beta v)}dr.$$

Putting this into $I_3'(\alpha,\beta)$ we obtain

$$I_{33}'(\alpha,\beta)=\frac{2^{12}\hat{w}(0)}{(\alpha+\beta)\log^{14}y_3}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1 y_3^{x(\beta-\alpha u)+y(\alpha-\beta v)}y_3^{-\alpha u-\beta v}\frac{(\log y_3)^{13}}{(12)!y_3^{-\alpha u-\beta v}}$$
$$\times(1-r)^{12}(x+r)^2P_3^{(6)}((1-u)(x+r))(y+r)^2$$
$$\times P_3^{(6)}((1-v)(y+r))y_3^{r(-\alpha u-\beta v)}drdudv\Bigg)\Bigg|_{x=y=0}$$
$$=\frac{2^{12}\hat{w}(0)}{12!(\alpha+\beta)\log y_3}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1 y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))}(1-r)^{12}$$
$$\times(x+r)^2(y+r)^2P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))drdudv\Bigg)\Bigg|_{x=y=0}.$$

To form the full $I_{33}(\alpha,\beta)$, recall that, as we discussed earlier, we need to add I_{33}' and I_{33}'', where I_{33}'' is formed by taking I_{33}', switching α and $-\beta$, and multiplying by $T^{-\alpha-\beta}$. Letting

$$U(\alpha,\beta)=\frac{y_3^{\beta(x-v(y+r))+\alpha(y-u(x+r))}-T^{-\alpha-\beta}y_3^{-\alpha(x-v(y+r))-\beta(y-u(x+r))}}{\alpha+\beta}$$

we then have

$$I_{33}(\alpha,\beta)=\frac{2^{12}\hat{w}(0)}{12!\log y_3}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1 y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))}(1-r)^{12}(x+r)^2$$
$$\times(y+r)^2U(\alpha,\beta)P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))drdudv\Bigg)\Bigg|_{x=y=0}$$
$$+O(TL^{-1+\varepsilon}).$$

Now write

$$U(\alpha,\beta) = y_3^{\beta(x-v(y+r))+\alpha(y-u(x+r))}\frac{1-\left(Ty_3^{x+y-v(y+r)-u(x+r)}\right)^{-\alpha-\beta}}{\alpha+\beta},$$

and use the integral formula

$$\frac{1-z^{-\alpha-\beta}}{\alpha+\beta} = \log z\int_0^1 z^{-t(\alpha+\beta)}dt,$$

as well as $y_3 = T^{\theta_3}$ so that

$$\begin{aligned}I_{33}(\alpha,\beta) = \frac{2^{12}}{12!}\hat{w}(0)\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1\int_0^1 &(1-r)^{12}y_3^{\beta(x-v(r+y))+\alpha(y-u(x+r))}\\ &\times\left(\frac{1}{\theta_3}+(x+y-v(y+r)-u(x+r))\right)(x+r)^2(y+r)^2\\ &\times P_3^{(6)}((1-u)(x+r))P_3^{(6)}((1-v)(y+r))\\ &\times(Ty_3^{x+y-v(y+r)-u(x+r)})^{-t(\alpha+\beta)}dtdrdudv\Bigg)\Bigg|_{x=y=0}.\end{aligned}$$

Hence this proves Lemma 5.5.

5.6 Proof of Proposition 5.6

Inserting the relevant definitions of the mollifiers in the mean value integral yields

$$\begin{aligned}I_{42}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\chi(\tfrac{1}{2}+it)\\ &\quad\times\sum_{ab\le y_2}\frac{\mu_2(a)}{a^{1/2+it}b^{1/2-it}}P_2[ab]\sum_{c\le y_4}\frac{\mu(c)}{c^{1/2-it}}\sum_{k=2}^{K}\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]dt\\ &=\sum_{k=2}^{K}\sum_{ab\le y_2}\sum_{c\le y_4}\frac{\mu_2(a)\mu(c)}{(abc)^{1/2}}P_2[ab]\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]J_{42},\end{aligned}$$

where

$$J_{42} = \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\chi(\tfrac{1}{2}+it)\left(\frac{a}{bc}\right)^{-it}dt.$$

This integral was evaluated in [BCY11, eq. (5.7)] and once we apply Lemma 4.1 of [BCY11] it is given by

$$J_{42} = \sum_{n=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(n)}{n^{1/2}}e^{-n/T^3}\int_{-\infty}^{\infty}w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{an}{bc}\right)^{-it}dt + O(T^{\varepsilon}).$$

Therefore, when we insert (5.25) in I_{42} we have

$$I_{42}(\alpha,\beta) = \sum_{k=2}^{K}\sum_{n=1}^{\infty}\sum_{ab\le y_2}\sum_{c\le y_4}\frac{\mu_2(a)\mu(c)\sigma_{\alpha,-\beta}(n)}{(abcn)^{1/2}}e^{-n/T^3}P_2[ab]\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c]$$

$$\times\, \widehat{w_0}\left(\frac{1}{2\pi}\log\frac{an}{bc}\right) + O(T^{(\theta_2+\theta_4)/2+\varepsilon}).$$

5.6.1 Off diagonal terms ($an \neq bc$):

Since $c \leq y_4$, then the sum satisfies

$$\sum_{p_1\cdots p_k|c} \frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c] \ll d(c).$$

The off-diagonal terms are given by

$$\begin{aligned} C_{42}(\alpha,\beta) = &\sum_{l=1}^{\infty}\sum_{bc\leq y_1}\sum_{\substack{f\leq y_4\\ bl\neq cf}} \frac{\mu_2(b)\mu(f)\sigma_{\alpha,-\beta}(l)}{(bcfl)^{1/2}} e^{-l/T^3} P_2[bc] \\ &\times \sum_{k=2}^{K}\sum_{p_1\cdots p_k|f} \frac{\log p_1\ldots\log p_k}{\log^k y_4}\widetilde{P}_k[f]\int_{-\infty}^{\infty} w_0(t)\left(\frac{bl}{cf}\right)^{-it} dt. \end{aligned}$$

In [BCY11] it is shown that

$$\widehat{w_0}(\tfrac{1}{2\pi}\log x) \ll_B \frac{T}{(1+\frac{1}{2\pi}\frac{T}{L}\log x)^B}, \tag{5.40}$$

for any $B \geq 0$. Let us split the above into

$$C_{42} = C'_{42} + C''_{42}, \quad \text{where} \quad C'_{42} = \sum_{1\leq l\leq T^4} \quad \text{and} \quad C''_{42} = \sum_{l>T^4}.$$

We have the following bound for C''_{42}

$$\begin{aligned} C''_{42} &\ll \sum_{l>T^4}\sum_{bc\leq y_2}\sum_{\substack{f\leq y_4\\ bl\neq cf}} \frac{|\mu_2(b)||\mu(f)|\sigma_{\alpha,-\beta}(l)d(f)}{(bcfl)^{1/2}} e^{-l/T^3}\int_{-\infty}^{\infty} w_0(t)dt \\ &\ll_\varepsilon \sum_{l>T^4}\frac{l^\varepsilon}{l^{1/2}}e^{-l/T^3}T^\varepsilon\left(\sum_{bc\leq y_2}\frac{1}{(bc)^{1/2}}\right)\left(\sum_{f\leq y_4}\frac{1}{f^{1/2}}\right) \\ &\ll_\varepsilon T^{(\theta_2+\theta_4)/2+2\varepsilon}\sum_{l>T^4}\frac{1}{l^{1/2-\varepsilon}}e^{-l/T^3} \\ &\ll_\varepsilon T^{\frac{3}{2}+\frac{1}{2}(\theta_2+\theta_4)+5\varepsilon}e^{-T}. \end{aligned}$$

We now assume that $\theta_2+\theta_4 < 1$. Fix δ_0 such that $0 < \delta_0 < 1-\theta_2-\theta_4$. For any $1 \leq l \leq T^4$, $1 \leq f \leq T^4$, $1 \leq f \leq y_4$ and any $b, c \geq 1$ such that $bc \geq y_2$ for which $cl \neq bf$ we have

$$\left|\frac{bl}{cf}-1\right| \geq \frac{1}{cf} \geq \frac{1}{y_2y_4} = \frac{1}{T^{\theta_2+\theta_4}} > \frac{1}{T^{1-\delta}}.$$

Therefore, we can write

$$\left|\log\frac{bl}{cf}\right| \geq \frac{1}{2T^{1-\delta_0}}.$$

Then by (5.40) with $B = \frac{2014}{\delta_0}$ we have, uniformly for all b, c, l and f in the above ranges, that

$$\int_{-\infty}^{\infty} w_0(t)\left(\frac{bl}{cf}\right)^{-it} dt \ll_{\delta_0} \frac{T}{(1+\frac{1}{2\pi}\frac{T}{L}\log x)^{2014/\delta_0}} \ll_{\delta_0} \frac{(T\log T)^{2014/\delta_0}}{T^{2014/\delta_0}} \ll_{\delta_0} \frac{1}{T^{2012}}.$$

We now use this to bound C'_{42} as follows

$$\begin{aligned} C'_{42} &\ll \sum_{1\le l\le T^4}\sum_{bc\le y_2}\sum_{\substack{f\le y_4\\ bl\ne cf}} \frac{|\mu_2(b)||\mu(f)|\sigma_{\alpha,-\beta}(l)d(f)}{(bcfl)^{1/2}} e^{-l/T^3}\left|\int_{-\infty}^{\infty} w_0(t)\left(\frac{bl}{cf}\right)^{-it} dt\right| \\ &\ll \sum_{1\le l\le T^4}\sum_{bc\le y_2}\sum_{\substack{f\le y_4\\ bl\ne cf}} \frac{T^{\varepsilon}}{(bcfl)^{1/2}}\frac{1}{T^{2012}} \ll \frac{T^{2\varepsilon+2+\frac{1}{2}(\theta_2+\theta_4)}}{T^{2012}} \ll \frac{1}{T^{2009}}. \end{aligned}$$

This shows that the off-diagonal terms get absorbed in the error term and do not contribute to our final answer.

5.6.2 Main term ($an = bc$):

From (5.23) and (5.24) we have

$$\begin{aligned} I_{42}(\alpha,\beta) = \widehat{w_0}(0)\sum_{k=2}^{K}\sum_{i,j} &\frac{a_i\widetilde{a}_{k,j}i!j!}{\log^i y_2 \log^{j+k} y_4}\left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} T^{3z}\Gamma(z) y_2^{z_1} y_4^{z_2} \\ &\times \sum_{an=bc} \frac{\mu_2(a)\mu(c)\sigma_{\alpha,-\beta}(n)}{(ab)^{1/2+z_1}c^{1/2+z_2}n^{1/2+z}} \sum_{p_1\cdots p_k | c} \log p_1\cdots\log p_k \frac{dz dz_1 dz_2}{z_1^{i+1}z_2^{j+1}} \\ &+ O(T^{1-\varepsilon}). \end{aligned} \tag{5.41}$$

Let us define

$$S_k = S_{k,\alpha,\beta}(z,z_1,z_2) = \sum_{an=bc} \frac{\mu_2(a)\mu(c)\sigma_{\alpha,-\beta}(n)}{(ab)^{1/2+z_1}c^{1/2+z_2}n^{1/2+z}} \sum_{p_1\cdots p_k | c} \log p_1\cdots\log p_k.$$

We begin by swapping the order of the sum so that

$$\begin{aligned} S_k &= (-1)^k \sum_{\substack{p_i\ne p_j\\ i<j}} \log p_1\ldots\log p_k \sum_{\substack{cl=bp_1\ldots p_k d\\ (d,p_1\ldots p_k)=1}} \frac{\mu_2(b)\mu(f)\sigma_{\alpha,-\beta}(l)}{(bc)^{1/2+z_1}d^{1/2+z_2}l^{1/2+z}} \frac{1}{(p_1\ldots p_k)^{1/2+z_2}} \\ &= (-1)^k \sum_{\substack{p_i\ne p_j\\ i<j}} \frac{\log p_1\ldots\log p_k}{(p_1\ldots p_k)^{1/2+z_2}} \sum_{\substack{b,c,d,f,l\\ (d,p_1\ldots p_k)=1}} \frac{\mu_2(b)\mu(f)}{(bc)^{1/2+z_1}d^{1/2+z_2}x^{1/2+\alpha+z}y^{1/2-\beta+z}}. \end{aligned}$$

As usual, let $\nu_p(n)$ denote the number of different prime factors of n. To simplify the expressions that will take place shortly, we simplify this notation to $\nu_p(n) = n'$. With this in mind, the above becomes

$$\begin{aligned}S_k &= (-1)^k \sum_{\substack{p_i\neq p_j\\ i<j}} \frac{\log p_1 \dots \log p_k}{(p_1\dots p_k)^{1/2+z_2}}\\ &\quad\times \prod_{p\in\{p_1,\cdots,p_k\}} \left(\sum_{b'+x'+y'=c'+1} \frac{\mu_2(p^{b'})}{(p^{b'}p^{c'})^{1/2+z_1}(p^{x'})^{1/2+\alpha+z}(p^{y'})^{1/2-\beta+z}}\right)\\ &\quad\times \prod_{p\notin\{p_1,\cdots,p_k\}} \left(\sum_{b'+x'+y'=c'+d'} \frac{\mu_2(p^{b'})\mu(p^{d'})}{(p^{b'}p^{c'})^{1/2+z_1}(p^{d'})^{1/2+z_2}(p^{x'})^{1/2+\alpha+z}(p^{y'})^{1/2-\beta+z}}\right)\end{aligned} \tag{5.42}$$

$$\begin{aligned}&= (-1)^k \sum_{\substack{p_i\neq p_j\\ i<j}} \frac{\log p_1 \dots \log p_k}{(p_1\dots p_k)^{1/2+z_2}} \frac{\Pi_1(k,\alpha,\beta)}{\Pi_2(k,\alpha,\beta)}\\ &\times \prod_p \bigg(1 + \frac{1}{p^{1+\alpha+z_1+z}} + \frac{1}{p^{1-\beta+z_1+z}} - \frac{1}{p^{1+\alpha+z_2+z}} - \frac{1}{p^{1-\beta+z_2+z}}\\ &\qquad\qquad - \frac{2}{p^{1+2z_1}} + \frac{2}{p^{1+z_1+z_2}} + O(p^{-2})\bigg),\end{aligned}$$

where

$$\Pi_1(k,\alpha,\beta) = \prod_{p\in\{p_1,\cdots,p_k\}} \left(\frac{1}{p^{1+\alpha+z+z_2}} + \frac{1}{p^{1-\beta+z+z_2}} - \frac{2}{p^{1+z_1+z_2}} + O(p^{-2})\right),$$

and

$$\begin{aligned}\Pi_2(k,\alpha,\beta) = \prod_{p\in\{p_1,\cdots,p_k\}} \bigg(1 &+ \frac{1}{p^{1+\alpha+z_1+z}} + \frac{1}{p^{1-\beta+z_1+z}} - \frac{1}{p^{1+\alpha+z_2+z}}\\ &- \frac{1}{p^{1-\beta+z_2+z}} - \frac{2}{p^{1+2z_1}} + \frac{2}{p^{1+z_1+z_2}} + O(p^{-2})\bigg).\end{aligned}$$

This reduces the expression for S_k to the more tractable

$$\begin{aligned}S_k &= \frac{\zeta(1+\alpha+z_1+z)\zeta(1-\beta+z_1+z)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2+z)\zeta(1-\beta+z_2+z)\zeta^2(1+2z_1)} A(\alpha,\beta,z,z_1,z_2)\\ &\quad\times(-1)^k \sum_{\substack{p_i\neq p_j\\ i<j}} \log p_1 \dots \log p_k\\ &\quad\times \prod_{p\in\{p_1,\cdots,p_k\}} \frac{E(p)+O(p^{-2})}{1+\frac{1}{p^{1+\alpha+z_1+z}}+\frac{1}{p^{1-\beta+z_2+z}}-\frac{2}{p^{1+2z_1}}-E(p)+O(p^{-2})},\end{aligned} \tag{5.43}$$

where

$$E(p) = \frac{1}{p^{1+\alpha+z+z_2}} + \frac{1}{p^{1-\beta+z+z_2}} - \frac{2}{p^{1+z_1+z_2}}.$$

By comparing (5.42) and (5.43) we see that

$$\frac{\zeta(1+\alpha+z_1+z)\zeta(1-\beta+z_1+z)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2+z)\zeta(1-\beta+z_2+z)\zeta^2(1+2z_1)} A(\alpha,\beta,z,z_1,z_2)$$

$$= \prod_p \Bigg(\sum_{b'+x'+y'=c'+d'} \frac{\mu_2(p^{b'})\mu(p^{d'})}{(p^{b'}p^{c'})^{1/2+z_1}(p^{d'})^{1/2+z_2}(p^{x'})^{1/2+\alpha+z}(p^{y'})^{1/2-\beta+z}} \Bigg). \tag{5.44}$$

Reverting the p-adic analysis on the right-hand side of (5.44) one arrives at

$$\begin{aligned}
&\frac{\zeta(1+\alpha+z_1+z)\zeta(1-\beta+z_1+z)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2+z)\zeta(1-\beta+z_2+z)\zeta^2(1+2z_1)} A(\alpha,\beta,z,z_1,z_2) \\
&= \prod_{bxy=cd} \frac{\mu_2(b)\mu(d)}{(bc)^{1/2+z_1}(d)^{1/2+z_2}(x)^{1/2+\alpha+z}(y)^{1/2-\beta+z}} \\
&= \prod_{bl=cd} \frac{\mu_2(b)\mu(d)\sigma_{\alpha,-\beta}(l)}{(bc)^{1/2+z_1}d^{1/2+z_2}l^{1/2+z}},
\end{aligned} \tag{5.45}$$

where in the ultimate step, we have used the definition of $\sigma_{\alpha,-\beta}(l)$. Using [BCY11, §5.6], we can conclude that $A(\alpha,\beta,z,z,z) = 1$ for all z. Let us denote the last line of (5.43) by H_k. Then

$$\begin{aligned}
H_k = (-1)^k &\sum_{\substack{p_i \neq p_j \\ i<j}} \prod_{p\in\{p_1,\cdots,p_k\}} (\log p)(E(p)+O(p^{-2})) \\
&\times \left(1 + E(p) - \frac{1}{p^{1+\alpha+z_1+z}} - \frac{1}{p^{1-\beta+z_2+z}} + \frac{2}{p^{1+2z_1}} + O(p^{-2})\right) \\
= (-1)^k &\sum_{\substack{p_i \neq p_j \\ i<j}} \prod_{p\in\{p_1,\cdots,p_k\}} \left(E(p)\log p + O\left(\frac{\log p}{p^2}\right)\right).
\end{aligned}$$

Next, we use the principle of inclusion-exclusion to write

$$H_k = (-1)^k \left(\sum_p E(p)\log p + O\left(\frac{\log p}{p}\right)\right)^k + \sum_p B(p),$$

where

$$B(p) \ll_{\alpha,\beta,z,z_1,z_2} \frac{1}{p^2}.$$

The final step is to identify sums over p containing $\log p$ with their analytic counterparts in terms of logarithmic derivatives of the zeta function by the use of

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)^{-1},$$

to see that

$$\begin{aligned}
H_k &= \left(-\frac{\zeta'}{\zeta}(1+\alpha+z+z_2) - \frac{\zeta'}{\zeta}(1-\beta+z_2+z) + 2\frac{\zeta'}{\zeta}(1+z+z_2) + O(\alpha,\beta,z,z_1,z_2)\right)^k \\
&\quad + D(\alpha,\beta,z,z_1,z_2) \\
&= U^k + \sum_{m=0}^{k-1} U^m B_m(\alpha,\beta,z,z_1,z_2),
\end{aligned} \tag{5.46}$$

where

$$U := 2\frac{\zeta'}{\zeta}(1+z_1+z_2) - \frac{\zeta'}{\zeta}(1+\alpha+z+z_2) - \frac{\zeta'}{\zeta}(1-\beta+z+z_2)$$

and

$$B_m(\alpha,\beta,z,z_1,z_2) \ll_{\alpha,\beta,z,z_1,z_2} \sum_p \frac{\log p}{p^2}.$$

All of these terms are analytic in a larger region, thus we need only be concerned with U^k. Next, we move the lines of integration to $\mathrm{Re}(z) = -\delta+\varepsilon$ and $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = \delta$. By deforming the contours like this, we cross the simple pole at $z = 0$ of $\Gamma(z)$. The integral on $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = \delta$, and $\mathrm{Re}(z) = -\delta+\varepsilon$ can be bounded by

$$\left|\widehat{w_0}(0)\frac{y_2^{\delta_0}y_4^{\delta_0}}{T^{3\delta_0}}\right|T^{3\varepsilon} \ll T^{1-\varepsilon}.$$

Hence

$$I_{42}(\alpha,\beta) = \widehat{w_0}(0)\sum_{k=2}^{K}\sum_{i,j}\frac{a_i\widetilde{a}_{k,j}\,i!j!}{\log^i y_2 \log^{j+k} y_4}K_{42} + O(T^{1-\varepsilon}),$$

where

$$\begin{aligned} K_{42} = &\left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} y_2^{z_1}y_4^{z_2}\frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)\zeta^2(1+2z_1)}A(\alpha,\beta,0,z_1,z_2) \\ &\times\left(2\frac{\zeta'}{\zeta}(1+z_1+z_2)-\frac{\zeta'}{\zeta}(1+\alpha+z_2)-\frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^k\frac{dz_1dz_2}{z_1^{i+1}z_2^{j+1}}. \end{aligned}$$

Let K'_{42} be the same integral as K_{42} but with $A(\alpha,\beta,z,z_1,z_2)$ replaced by $A(\alpha,\beta,0,0,0) = (-1)^k$. Then, just as before, $K'_{42} = K_{42} + O(L^{i+j-1})$. We wish to separate the variables z_1 and z_2 by the use of a suitable Dirichlet series. Let us define the term involving ζ's in the integrand of K_{42} by Π_{42}. Using the multinomial theorem we have

$$\begin{aligned} \Pi_{42} = &\frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)\zeta^2(1+z_1+z_2)}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)\zeta^2(1+2z_1)} \\ &\qquad\times\left(2\frac{\zeta'}{\zeta}(1+z_1+z_2)-\frac{\zeta'}{\zeta}(1+\alpha+z_2)-\frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^k \\ = &(-1)^k k!\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}}{l_1!l_2!l_3!}\sum_{n=1}^{\infty}\frac{(d*\Lambda^{*l_1})(n)}{n^{1+z_1+z_2}}\frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)}{\zeta^2(1+2z_1)} \\ &\qquad\times\frac{1}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)}\left(\frac{\zeta'}{\zeta}(1+\alpha+z_2)\right)^{l_2}\left(\frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^{l_3}, \end{aligned}$$

where we have used the Dirichlet convolution of

$$\zeta^2(s) = \sum_{n=1}^{\infty}\frac{d(n)}{n^s} \quad \text{and} \quad -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},$$

for $\mathrm{Re}(s) > 1$ and where Λ^{*l_1} stands for convolving $\Lambda*\cdots*\Lambda$ exactly l_1 times. Hence, we get the splitting

$$K'_{42} = \frac{k!}{\log^k y_4}\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}}{l_1!l_2!l_3!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{l_1})(n)}{n}K_1K_2(l_2,l_3) + O(L^{i+j-1}),$$

where

$$K_1 = \frac{1}{2\pi i}\int_{(\delta)}\left(\frac{y_2}{n}\right)^{z_1}\frac{\zeta(1+\alpha+z_1)\zeta(1-\beta+z_1)}{\zeta^2(1+2z_1)}\frac{dz_1}{z_1^{i+1}},$$

and

$$\begin{aligned}K_2(l_2,l_3) = \frac{1}{2\pi i}\int_{(\delta)}\left(\frac{y_4}{n}\right)^{z_2}&\frac{1}{\zeta(1+\alpha+z_2)\zeta(1-\beta+z_2)}\\ &\times\left(\frac{\zeta'}{\zeta}(1+\alpha+z_2)\right)^{l_2}\left(\frac{\zeta'}{\zeta}(1-\beta+z_2)\right)^{l_3}\frac{dz_2}{z_2^{j+1}}.\end{aligned} \tag{5.47}$$

From [BCY11, eq. (5.41)] we have

$$K_1 = \frac{4(\log(y_2/n))^i}{(i-2)!}\iint_{a+b\le 1}(1-a-b)^{i-2}\left(\frac{y_2}{n}\right)^{-a\alpha+b\beta}dadb + O(L^{i-1}).$$

By the Laurent series expansion around $s=1$ of the logarithmic derivative of $\zeta(s)$ we have that

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1}+\gamma+O(|s-1|). \tag{5.48}$$

Now we will compute the following contour integrations for different choices of l_2 and l_3. This process yields the following cases.

1. If $l_2 = l_3 = 0$,

$$\begin{aligned}\frac{1}{2\pi i}\oint q^{z_2}(\alpha+z_2)(-\beta+z_2)\frac{dz_2}{z_2^{j+1}} &= \frac{d^2}{dxdy}e^{\alpha x-\beta y}\frac{1}{2\pi i}\oint (qe^{x+y})^{z_2}\frac{dz_2}{z_2^{j+1}}\Bigg|_{x=y=0}\\ &= \frac{1}{j!}\frac{d^2}{dxdy}\left[e^{\alpha x-\beta y}\left(x+y+\log\frac{y_4}{n}\right)^j\right]_{x=y=0}.\end{aligned} \tag{5.49}$$

2. If $l_2 = 1$ and $l_3 = 0$,

$$\begin{aligned}-\frac{1}{2\pi i}\oint q^{z_2}(-\beta+z_2)\frac{dz_2}{z_2^{j+1}} &= -\frac{d}{dy}e^{-\beta y}\frac{1}{2\pi i}\oint (qe^{y})^{z_2}\frac{dz_2}{z_2^{j+1}}\Bigg|_{y=0}\\ &= -\frac{1}{j!}\frac{d}{dy}\left[e^{-\beta y}\left(y+\log\frac{y_4}{n}\right)^j\right]_{y=0}.\end{aligned} \tag{5.50}$$

3. By symmetry, if $l_2 = 0$ and $l_3 = 1$, then

$$-\frac{1}{2\pi i}\oint q^{z_2}(\alpha+z_2)\frac{dz_2}{z_2^{j+1}} = -\frac{1}{j!}\frac{d}{dy}\left[e^{\alpha y}\left(y+\log\frac{y_4}{n}\right)^j\right]_{y=0}. \tag{5.51}$$

4. If $l_2 = l_3 = 1$,

$$\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{dz_2}{z_2^{j+1}} = \frac{1}{j!}\log^j\frac{y_4}{n}.$$

5. If $l_2 = 1$ and $l_3 \geq 2$,

$$\begin{aligned}
&(-1)^{1+l_3}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{1}{(-\beta+z_2)^{l_3-1}}\frac{dz_2}{z_2^{j+1}}\\
&=-\frac{1}{(l_3-2)!}\int_{1/q}^{1}t^{-\beta-1}\log^{l_3-2}t\frac{1}{2\pi i}\oint(qt)^{z_2}\frac{dz_2}{z_2^{j+1}}dt\\
&=-\frac{1}{j!(l_3-2)!}\int_{n/y_4}^{1}t^{-\beta-1}\log^{j}\left(\frac{y_4}{n}t\right)\log^{l_3-2}tdt\\
&=-\frac{(-1)^{l_3-2}(\log(y_4/n))^{j+l_3-1}}{j!(l_3-2)!}\int_0^1(1-b)^j\left(\frac{y_4}{n}\right)^{b\beta}b^{l_3-2}db. \qquad (5.52)
\end{aligned}$$

6. Again, by symmetry, if $l_2 \geq 2$ and $l_3 = 1$, then

$$\begin{aligned}
&(-1)^{1+l_2}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{1}{(\alpha+z_2)^{l_2-1}}\frac{dz_2}{z_2^{j+1}}\\
&=-\frac{(-1)^{l_2-2}(\log(y_4/n))^{j+l_2-1}}{j!(l_2-2)!}\int_0^1(1-b)^j\left(\frac{y_4}{n}\right)^{-b\alpha}b^{l_2-2}db. \qquad (5.53)
\end{aligned}$$

7. If $l_2 = 0$ and $l_3 \geq 2$,

$$\begin{aligned}
&(-1)^{l_3}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{\alpha+z_2}{(-\beta+z_2)^{l_3-1}}\frac{dz_2}{z_2^{j+1}}\\
&=\frac{(-1)^{l_3}}{j!(l_3-2)!}\frac{d}{dx}\left(x+\log\frac{y_4}{m}\right)^{j+l_3-1}e^{\alpha x}\int_0^1c^{l_3-2}(1-c)^j\left(\frac{y_4}{m}\right)^{\beta c}e^{\beta cx}dc\bigg|_{x=0}. \qquad (5.54)
\end{aligned}$$

8. If $l_2 \geq 2$ and $l_3 = 0$,

$$\begin{aligned}
&(-1)^{l_2}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{-\beta+z_2}{(\alpha+z_2)^{l_2-1}}\frac{dz_2}{z_2^{j+1}}+O(L^{j-3-l_2})\\
&=\frac{(-1)^{l_2}}{j!(l_2-2)!}\frac{d}{dx}\left(x+\log\frac{y_4}{n}\right)^{j+l_2-1}e^{-\beta x}\int_0^1c^{l_2-2}(1-c)^j\left(\frac{y_4}{n}\right)^{-\alpha c}e^{-\alpha cx}dc\bigg|_{x=0}. \qquad (5.55)
\end{aligned}$$

9. Finally, if $l_2 \geq 2$ and $l_3 \geq 2$,

$$\begin{aligned}
&(-1)^{l_2+l_3}\frac{1}{2\pi i}\oint\left(\frac{y_4}{n}\right)^{z_2}\frac{1}{(\alpha+z_2)^{l_2-1}}\frac{1}{(-\beta+z_2)^{l_3-1}}\frac{dz_2}{z_2^{j+1}}\\
&=(-1)^{l_2+l_3}\frac{(-1)^{2-l_2}}{(l_2-2)!}\frac{(-1)^{2-l_3}}{(l_3-2)!}\int_{1/q}^{1}\int_{1/(qr)}^{1}r^{\alpha-1}t^{-\beta-1}\log^{l_2-2}r\log^{l_3-2}t\\
&\qquad\times\frac{1}{2\pi i}\oint(qrt)^{z_2}\frac{dz_2}{z_2^{j+1}}dtdr\\
&=\frac{1}{j!(l_2-2)!(l_3-2)!}\int_{1/q}^{1}\int_{1/(qr)}^{1}r^{\alpha-1}t^{-\beta-1}\log^{l_2-2}r\log^{l_3-2}t\log\left(rt\frac{y_4}{n}\right)^j dtdr\\
&=\frac{(-1)^{l_2+l_3}(\log(y_4/n))^{j+l_2+l_3-2}}{j!(l_2-2)!(l_3-2)!}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}(1-a-b)^j\left(\frac{y_4}{n}\right)^{-a\alpha+b\beta}
\end{aligned}$$

$$\times\, a^{l_2-2}b^{l_3-2}dadb. \tag{5.56}$$

In the last step we used the substitutions $r = q^{-a}$ and $t = q^{-b}$.

Going back to $I_{42}(\alpha,\beta)$, we now have to perform the sums over i and j and then insert them back into

$$I_{42}(\alpha,\beta) = \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}}{l_1!l_2!l_3!}\sum_{n\leq\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}$$
$$\times\sum_i K_1\frac{a_i i!}{\log^i y_2}\sum_j\frac{\widetilde{a}_{k,j}j!}{\log^j y_4}K_2(l_2,l_3)+O(TL^{-1+\varepsilon}). \tag{5.57}$$

Since $\theta_4<\theta_2$, we will now use $\min(y_2,y_4)=y_4$. From [BCY11, §5.5] we find

$$\sum_i\frac{a_i i!}{\log^i y_2}K_1 = 4\frac{(\log(y_2/n))^2}{(\log y_2)^2}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\left(\frac{y_2}{n}\right)^{-a\alpha+b\beta}P_2''\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right)dadb$$
$$+O(L^{-1}).$$

For the j-sum, we need to consider each case separately.

5.6.2.1 The case $l_2=l_3=0$

In this case, from (5.32), (5.48), (5.49), and a similar argument to that of of Lemma 5.16 we have

$$\sum_j^{(0,0)} = \sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(0,0) = \frac{d^2}{dxdy}e^{\alpha x-\beta y}\sum_j\widetilde{a}_{j,k}\left(\frac{x+y}{\log y_4}+\frac{\log(y_4/n)}{\log y_4}\right)^j\Bigg|_{x=y=0}+O(L^{-3})$$
$$=\frac{1}{(\log y_4)^2}\frac{d^2}{dxdy}y_4^{\alpha x-\beta y}\widetilde{P}_k\left(x+y+\frac{\log(y_4/n)}{\log y_4}\right)\Bigg|_{x=y=0}+O(L^{-3}).$$

Inserting this expression in (5.57) yields

$$I_{42}^{(0,0)}(\alpha,\beta) = \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1=k}\frac{2^{l_1}}{l_1!}\sum_{n\leq\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(0,0)}+O(TL^{-1+\varepsilon})$$
$$=\widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\frac{2^k}{k!}\sum_{n\leq y_4}\frac{(d*\Lambda^{*k})(n)}{n}\sum_i\sum_j^{(0,0)}+O(TL^{-1+\varepsilon})$$
$$=\frac{4\widehat{w_0}(0)}{(\log y_4)^2}\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\frac{2^k}{k!}\frac{d^2}{dxdy}\Bigg[y_2^{\alpha x-\beta y}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\sum_{n\leq y_4}\frac{(d*\Lambda^{*k})(n)}{n^{1-a\alpha+b\beta}}y_2^{-a\alpha+b\beta}$$
$$\times P_2''\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right)\left(\frac{\log(y_2/n)}{\log y_2}\right)^2$$
$$\times\widetilde{P}_k\left(x+y+\frac{\log(y_4/n)}{\log y_4}\right)dadb\Bigg]_{x=y=0}+O(TL^{-1+\varepsilon})$$
$$=4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^k}{(1+k)!}\frac{d^2}{dxdy}\Bigg[y_2^{\alpha x-\beta y}\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2$$

$$
\times (1-u)^{1+k}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta} P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)
\times \widetilde{P}_k(x+y+u)dudadb\Bigg]_{x=y=0} + O(TL^{-1+\varepsilon}),
$$

where we have applied Lemma 5.15 with $k=2$, $l=k$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(x+y+u)$.

5.6.2.2 The case $l_2=1$, $l_3=0$

In this case, from (5.32), (5.48), (5.50), and a similar argument to that of Lemma 5.16 we have

$$
\begin{aligned}
\sum_j^{(1,0)} &= \sum_j \frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(1,0) = -\frac{d}{dy}e^{-\beta y}\sum_j \widetilde{a}_{j,k}\left(\frac{y}{\log y_4}+\frac{\log(y_4/n)}{\log y_4}\right)^j\Bigg|_{y=0} + O(L^{-4})\\
&= -\frac{1}{\log y_4}\frac{d}{dy}y_4^{-\beta y}\widetilde{P}_k\left(y+\frac{\log(y_4/n)}{\log y_4}\right)\Bigg|_{y=0} + O(L^{-4}).
\end{aligned}
$$

By an analogue argument as in the previous case

$$
\begin{aligned}
I_{42}^{(1,0)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+1=k}\frac{2^{l_1}}{l_1!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(1,0)} + O(TL^{-1+\varepsilon})\\
&= -4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^{k-1}}{(k-1)!}\frac{d}{dy}\Bigg[y_4^{-\beta y}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^k\\
&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(y+u)dudadb\Bigg]_{y=0}\\
&\quad + O(TL^{-1+\varepsilon}),
\end{aligned}
$$

where we have used $k=2$, $l=k-1$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(y+u)$ in Lemma 5.15.

5.6.2.3 The case $l_2=0$, $l_3=1$

In this case, from (5.32), (5.48), (5.51), and a similar argument to that of Lemma 5.16 we have

$$
\begin{aligned}
\sum_j^{(0,1)} &= \sum_j \frac{\tilde{a}_{k,j}j!}{\log^j y_4}K_2(0,1) = -\frac{d}{dx}e^{\alpha x}\sum_j \tilde{a}_{k,j}\left(\frac{x}{\log y_4}+\frac{\log(y_4/n)}{\log y_4}\right)^j\Bigg|_{x=0}\\
&= -\frac{1}{\log y_4}\frac{d}{dx}y_4^{\alpha x}\tilde{P}_k\left(x+\frac{\log(y_4/n)}{\log y_4}\right)\Bigg|_{x=0} + O(L^{-4}).
\end{aligned}
$$

Similarly

$$
I_{42}^{(0,1)}(\alpha,\beta) = \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+1=k}\frac{2^{l_1}}{l_1!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(0,1)} + O(TL^{-1+\varepsilon})
$$

$$
\begin{aligned}
&= -4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^{k-1}}{(k-1)!}\frac{d}{dx}\Bigg[y_4^{\alpha x}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^k\\
&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+u)dudadb\Bigg]_{x=0}\\
&\qquad+O(TL^{-1+\varepsilon}),
\end{aligned}
$$

by setting $k=2$, $l=k-1$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(x+u)$ in Lemma 5.15.

5.6.2.4 The case $l_2=1$, $l_3=1$

In this case, from (5.32), (5.48), and Cauchy's theorem we have

$$
\sum_j^{(1,1)} = \sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(1,1) = \sum_j\widetilde{a}_{j,k}\left(\frac{\log(y_4/n)}{\log y_4}\right)^j+O(L^{-5}) = \widetilde{P}_k\left(\frac{\log(y_4/n)}{\log y_4}\right)+O(L^{-5}).
$$

Hence

$$
\begin{aligned}
I_{42}^{(1,1)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+2=k}\frac{2^{l_1}}{l_1!}\sum_{n\le\min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(1,1)}+O(TL^{-1+\varepsilon})\\
&= 4\widehat{w_0}(0)\sum_{k=2}^{K}\frac{2^{k-2}k}{(k-2)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{k-1}\\
&\times\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(u)dudadb\\
&\qquad+O(TL^{-1+\varepsilon}),
\end{aligned}
$$

by setting $k=2$, $l=k-2$, $s=-a\alpha+b\beta$, $z=y_4$, $x=y_2$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=\widetilde{P}_k(u)$ in Lemma 5.15.

5.6.2.5 The case $l_2=1$, $l_3\ge 2$

In this case, from (5.32), (5.48), (5.52), and a similar argument to that of Lemma 5.16 we have

$$
\begin{aligned}
\sum_j^{(1,l_3)} &= \sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(1,l_3)\\
&= -\frac{(-1)^{l_3-2}(\log(y_4/n))^{l_3-1}}{(l_3-2)!}\int_0^1\sum_j\widetilde{a}_{j,k}\left((1-b)\frac{(\log(y_4/n))}{\log y_4}\right)^j\left(\frac{y_4}{n}\right)^{b\beta}b^{l_3-2}db\\
&\qquad+O(L^{-4-l_3})\\
&= -\frac{(-1)^{l_3-2}(\log(y_4/n))^{l_3-1}}{(l_3-2)!}\int_0^1\widetilde{P}_k\left((1-c)\frac{\log(y_4/n)}{\log y_4}\right)\left(\frac{y_4}{n}\right)^{c\beta}c^{l_3-2}dc\\
&\qquad+O(L^{-4-l_3}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I_{42}^{(1,\geq 2)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+1+l_3=k}\frac{2^{l_1}}{l_1!l_3!}\sum_{n\leq \min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(1,l_3)} \\
&\quad + O(TL^{-1+\varepsilon}) \\
&= -4\widehat{w_0}(0)\sum_{k=2}^{K}k!\sum_{l_1+1+l_3=k}\frac{2^{l_1}(-1)^{l_3-2}}{l_1!l_3!(1+l_1)!(l_3-2)!} \\
&\quad\times \iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^{1+l_1}u^{l_3-1}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta} \\
&\quad\times c^{l_3-2}y_4^{uc\beta}P_2''\left((1-a-b)\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)\right)\widetilde{P}_k((1-c)u)dudcdadb \\
&\quad + O(TL^{-1+\varepsilon}),
\end{aligned}
$$

by setting $k=2$, $l=l_1$, $z=y_4$, $x=y_2$, $s=-a\alpha+b\beta+c\beta$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=u^{l_3-1}\widetilde{P}_k((1-c)u)$ in Lemma 5.15.

5.6.2.6 The case $l_2\geq 2$, $l_3=1$

In this case, from (5.32), (5.48), (5.53), and a similar argument to that of Lemma 5.16 we have

$$
\begin{aligned}
\sum_j^{(l_2,1)} &= \sum_j\frac{\widetilde{a}_{j,k}j!}{\log^j y_4}K_2(l_2,1)+O(L^{-4-l_2}) \\
&= -\frac{(-1)^{l_2-2}(\log(y_4/n))^{l_2-1}}{(l_2-2)!}\int_0^1\widetilde{P}_k\left((1-c)\frac{\log(y_4/n)}{\log y_4}\right)\left(\frac{y_4}{n}\right)^{-c\alpha}c^{l_2-2}dc \\
&\quad + O(L^{-4-l_2}).
\end{aligned}
$$

Similarly one has

$$
\begin{aligned}
I_{42}^{(\geq 2,1)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2+1=k}\frac{2^{l_1}}{l_1!l_2!}\sum_{n\leq \min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(l_2,1)} \\
&\quad + O(TL^{-1+\varepsilon}) \\
&= -4\widehat{w_0}(0)\sum_{k=2}^{K}k!\sum_{l_1+l_2+1=k}\frac{2^{l_1}(-1)^{l_2-2}}{l_1!l_2!(1+l_1)!(l_2-2)!} \\
&\quad\times \iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2(1-u)^{1+l_1}u^{l_2-1}\left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta} \\
&\quad\times c^{l_2-2}y_4^{-uc\alpha}P_2''\left((1-a-b)\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)\right)\widetilde{P}_k((1-c)u)dudcdadb \\
&\quad + O(TL^{-1+\varepsilon}),
\end{aligned}
$$

by setting $k=2$, $l=l_1$, $z=y_4$, $x=y_2$, $s=-a\alpha+b\beta-c\alpha$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=u^{l_2-1}\widetilde{P}_k((1-c)u)$ in Lemma 5.15.

5.6.2.7 The case $l_2 = 0$, $l_3 \geq 2$

By a similar argument to that of Lemmas 5.17, 5.18, and using Lemma 5.14 together with equation (5.54) we have

$$\begin{aligned}
\sum_j^{(0,l_3)} &= \sum_j \frac{\tilde{a}_{k,j} j!}{\log^j y_4} K_2(0,l_3) + O(L^{-3-l_3}) \\
&= \sum_j \frac{\tilde{a}_{k,j} j!}{\log^j y_4} \frac{(-1)^{l_3}}{j!(l_3-2)!} \frac{d}{dx}\left(x + \log\frac{y_4}{n}\right)^{l_3+j-1} e^{\alpha x} \int_0^1 \left(\frac{y_4}{n}\right)^{c\beta} (1-c)^j c^{l_3-2} e^{c\beta x} dc\Bigg|_{x=0} \\
&+ O(L^{-3-l_3}) \\
&= \frac{(-1)^{l_3}}{(l_3-2)!} \log^{l_3-2} y_4 \frac{d}{dx}\Bigg[y_4^{\alpha x} \left(x + \frac{\log\frac{y_4}{n}}{\log y_4}\right)^{l_3-1} \\
&\qquad \times \int_0^1 \tilde{P}_k\left((1-c)\left(x + \frac{\log\frac{y_4}{n}}{\log y_4}\right)\right)\left(\frac{y_4}{n}\right)^{c\beta} c^{l_3-2} e^{c\beta x\log y_4} dc\Bigg]_{x=0} + O(L^{-3-l_3}),
\end{aligned}$$

after the change $y = x/\log y_4$. As done previously

$$\begin{aligned}
I_{42}^{(0,l_3)}(\alpha,\beta) &= \widehat{w_0}(0) \sum_{k=2}^K \frac{k!}{\log^k y_4} \sum_{l_1+l_3=k} \frac{2^{l_1}}{l_1! l_3!} \sum_{n \leqslant \min(y_2,y_4)} \frac{(d * \Lambda^{*l_1})(n)}{n} \sum_i \sum_j^{(0,l_3)} \\
&\quad + O(TL^{-1+\varepsilon}) \\
&= 4\widehat{w_0}(0) \sum_{k=2}^K \frac{k!}{\log^k y_4} \sum_{l_1+l_3=k} \frac{2^{l_1}}{l_1! l_3!} \frac{(-1)^{l_3} \log^{l_3-2} y_4}{(l_3-2)!} \frac{d}{dx}\Bigg[y_4^{\alpha x} \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \int_0^1 \\
&\quad \times y_2^{-a\alpha+b\beta} \sum_{n \leqslant y_4} \frac{(d * \Lambda^{*l_1})(n)}{n^{1-a\alpha+b\beta+c\beta}} \left(\frac{\log(y_2/n)}{\log y_2}\right)^2 P''_2\left((1-a-b)\frac{\log(y_2/n)}{\log y_2}\right) \\
&\quad \times \left(x + \frac{\log\frac{y_4}{n}}{\log y_4}\right)^{l_3-1} \tilde{P}_k\left((1-c)\left(x + \frac{\log\frac{y_4}{n}}{\log y_4}\right)\right) y_4^{c\beta(1+x)} c^{l_3-2} da db dc\Bigg]_{x=0} \\
&\quad + O(TL^{-1+\varepsilon}) \\
&= 4\widehat{w_0}(0) \sum_{k=2}^K k! \sum_{l_1+l_3=k} \frac{2^{l_1}(-1)^{l_3}}{l_1! l_3! (l_3-2)! (1+l_1)!} \frac{d}{dx}\Bigg[y_4^{\alpha x} \iint_{\substack{0 \le a+b \le 1 \\ a,b \ge 0}} \int_0^1 \int_0^1 \\
&\quad \times (1-u)^{1+l_1} (x+u)^{l_3-1} y_4^{c\beta(u+x)} \left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha-b\beta} \left(1 - (1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_3-2} \\
&\quad \times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \tilde{P}_k((1-c)(x+u)) da db dc du\Bigg]_{x=0} \\
&\quad + O(TL^{-1+\varepsilon}),
\end{aligned}$$

by setting $k = 2$, $l = l_1$, $z = y_4$, $x = y_2$, $s = -a\alpha + b\beta + c\beta$, $F(u) = u^2 P''_2((1-a-b)u)$ and $H(u) = (x+u)^{l_3-1}\tilde{P}_k((1-c)(x+u))$ in Lemma 5.15.

5.6.2.8 The case $l_2 \geq 2$, $l_3 = 0$

Again, by a similar argument to that of Lemmas 5.17, 5.18, and using Lemma 5.14 together with equation (5.55) we have

$$\begin{aligned}
\sum_j^{(l_2,0)} &= \sum_j \frac{\tilde{a}_{k,j} j!}{\log^j y_4} K_2(l_2,0) + O(L^{-3-l_2}) \\
&= \sum_j \frac{\tilde{a}_{k,j} j!}{\log^j y_4} \frac{(-1)^{l_2}}{j!(l_2-2)!} \frac{d}{dx}\left(x + \log \frac{y_4}{n}\right)^{j+l_2-1} \\
&\qquad \times e^{-\beta x} \int_0^1 \left(\frac{y_4}{n}\right)^{-\alpha c} (1-c)^j c^{l_2-2} e^{-\alpha c x} dc \bigg|_{x=0} + O(L^{-3-l_2}) \\
&= \frac{(-1)^{l_2}}{(l_2-2)!} \log^{l_2-2} y_4 \frac{d}{dx}\Bigg[y_4^{-\beta x} \left(x + \frac{\log \frac{y_4}{n}}{\log y_4}\right)^{l_2-1} \\
&\qquad \times \int_0^1 \tilde{P}_k\left((1-c)\left(x + \frac{\log \frac{y_4}{n}}{\log y_4}\right)\right) \left(\frac{y_4}{n}\right)^{-\alpha c} c^{l_2-2} y_4^{-\alpha c x} dc \Bigg]_{x=0} + O(L^{-3-l_2}),
\end{aligned}$$

after the change $y = x/\log y_4$. Likewise, one has

$$\begin{aligned}
I_{42}^{(l_2,0)}(\alpha,\beta) &= \widehat{w_0}(0) \sum_{k=2}^K \frac{k!}{\log^k y_4} \sum_{l_1+l_2=k} \frac{2^{l_1}}{l_1! l_2!} \sum_{n \leqslant \min(y_2,y_4)} \frac{(d * \Lambda^{*l_1})(n)}{n} \sum_i \sum_j^{(l_2,0)} \\
&\qquad + O(TL^{-1+\varepsilon}) \\
&= 4\widehat{w_0}(0) \sum_{k=2}^K \frac{k!}{\log^k y_4} \sum_{l_1+l_2=k} \frac{2^{l_1}}{l_1! l_2!} \frac{(-1)^{l_2} \log^{l_2-2} y_4}{(l_2-2)!} \frac{d}{dx}\Bigg[y_4^{-\beta x} \iint_{\substack{0 \leq a+b \leq 1 \\ a,b \geq 0}} \int_0^1 \\
&\qquad \times y_2^{-a\alpha+b\beta} \sum_{n \leqslant y_4} \frac{(d * \Lambda^{*l_1})(n)}{n^{1-a\alpha+b\beta-\alpha c}} \left(\frac{\log(y_2/n)}{\log y_2}\right)^2 P''_2\left((1-a-b) \frac{\log(y_2/n)}{\log y_2}\right) \\
&\qquad \times \left(x + \frac{\log \frac{y_4}{n}}{\log y_4}\right)^{l_2-1} \tilde{P}_k\left((1-c)\left(x + \frac{\log \frac{y_4}{n}}{\log y_4}\right)\right) y_4^{-\alpha c(1+x)} c^{l_2-2} da db dc \Bigg]_{x=0} \\
&\qquad + O(TL^{-1+\varepsilon}) \\
&= 4\widehat{w_0}(0) \sum_{k=2}^K k! \sum_{l_1+l_2=k} \frac{2^{l_1}}{l_1! l_2!} \frac{(-1)^{l_2}}{(l_2-2)!(1+l_1)!} \frac{d}{dx}\Bigg[y_4^{-\beta x} \iint_{\substack{0 \leq a+b \leq 1 \\ a,b \geq 0}} \int_0^1 \int_0^1 \\
&\qquad \times (1-u)^{1+l_1} (x+u)^{l_2-1} y_4^{-\alpha c(u+x)} \left(\frac{y_4^{1-u}}{y_2}\right)^{a\alpha - b\beta} \left(1 - (1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_2-2} \\
&\qquad \times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \tilde{P}_k((1-c)(x+u)) da db dc du \Bigg]_{x=0} \\
&\qquad + O(TL^{-1+\varepsilon}),
\end{aligned}$$

by the use of $k=2$, $l=l_1$, $z=y_4$, $x=y_2$, $s=-a\alpha+b\beta-\alpha c$, $F(u)=u^2 P''_2((1-a-b)u)$ and $H(u)=(x+u)^{l_3-1}\tilde{P}_k((1-c)(x+u))$ in Lemma 5.15.

5.6.2.9 The case $l_2 \geq 2$, $l_3 \geq 2$

Lastly, from (5.32), (5.48), and a similar argument to that of Lemma 5.16 we have

$$\begin{aligned}\sum_{j}^{(l_2,l_3)} &= \sum_j \frac{\widetilde{a}_{k,j} j!}{\log^j y_4} K_2(l_2,l_3) + O(L^{-3-l_2-l_3}) \\ &= \frac{(-1)^{l_2+l_3}(\log(y_4/n))^{l_2+l_3-2}}{(l_2-2)!(l_3-2)!} \\ &\quad \times \iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}} \widetilde{P}_k\left((1-a-b)\left(\frac{\log(y_4/n)}{\log y_4}\right)\right)\left(\frac{y_4}{n}\right)^{-a\alpha+b\beta} a^{l_2-l_2} b^{l_3-2} dadb \\ &\quad + O(L^{-3-l_2-l_3}).\end{aligned}$$

Finally,

$$\begin{aligned}I_{42}^{(l_2,l_3)}(\alpha,\beta) &= \widehat{w_0}(0)\sum_{k=2}^{K}\frac{k!}{\log^k y_4}\sum_{l_1+l_2+l_3=k}\frac{2^{l_1}}{l_1!l_2!l_3!}\sum_{n\leq \min(y_2,y_4)}\frac{(d*\Lambda^{*l_1})(n)}{n}\sum_i\sum_j^{(l_2,l_3)} \\ &\quad + O(TL^{-1+\varepsilon}) \\ &= 4\widehat{w_0}(0)\sum_{k=2}^{K} k! \sum_{l_1+l_2+l_3=k}\frac{2^{l_1}(-1)^{l_2+l_3}}{l_1!l_2!l_3!(1+l_1)!(l_2-2)!(l_3-2)!} \\ &\quad \times \iiiint_{\substack{0\leq a+b\leq 1\\ 0\leq g+h\leq 1\\ a,b,g,h\geq 0}}\int_0^1 (1-u)^{k+l-1}\left(\frac{y_4}{y_2}\right)^{a\alpha-b\beta}\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)^2 \\ &\quad \times P_2''\left((1-a-b)\left(1-(1-u)\frac{\log y_4}{\log y_2}\right)\right)\widetilde{P}_k((1-g-h)u) \\ &\quad \times y_4^{-a\alpha u-g\alpha u+b\beta u+h\beta u} u^{l_2+l_3-2} g^{l_2-2} h^{l_3-2} dudadbdgdh + O(TL^{-1+\varepsilon}),\end{aligned}$$

by setting $k=2$, $l=l_1$, $z=y_4$, $x=y_2$, $s=-a\alpha+b\beta-g\alpha+h\beta$, $F(u)=u^2P_2''((1-a-b)u)$ and $H(u)=u^{l_2+l_3-2}\widetilde{P}_k((1-g-h)u)$ in Lemma 5.15.

5.7 Proof of Proposition 5.7

We will first focus on the error terms. From [Con89, p. 11, Proposition] we can obtain the right order of magnitude of the error term for $I_{11}(\alpha,\beta,w)$ when $\theta_1 < 4/7-\varepsilon$. To see the error terms for $I_{14}(\alpha,\beta,w)$ and $I_{44}(\alpha,\beta,w)$, we will proceed as follows. First we set $\psi_1(s)=\sum_{n\leq y_1} b(n)n^{-s}$ and $\psi_4(s)=\sum_{m\leq y_4} c(n)n^{-s}$. We state our result following a similar style to that of Proposition of [Con89].

Proposition 5.19. *Let $\theta_1 < 4/7-\varepsilon$, $\theta_4 < 3/7-\varepsilon$, and $T/2 \leq w \leq T$. Then we have*

$$I_{14}(\alpha,\beta,w) = \frac{\sum(\beta,\alpha) - e^{-(\alpha+\beta)L}\sum(-\alpha,-\beta)}{\alpha+\beta} + O(L^{-1+\varepsilon}), \tag{5.58}$$

where

$$\sum(\beta,\alpha) := \sum_{\substack{n\leq y_1\\ m\leq y_4}} \frac{\mathfrak{b}(n)\mathfrak{c}(m)}{n^{1+\alpha}m^{1+\beta}}(n,m)^{1+\alpha+\beta}.$$

Proof. For the sake of brevity, we will follow the proof of Proposition of [Con89]. More precisely we will follow the steps starting from equation (50) and ending in equation (69). The only modification we need is that $b(h, P_1) = \mathfrak{b}(h)$ and $b(k, P_2) = \mathfrak{c}(k)$. By doing so, we arrive to the following step:

$$\mathcal{M}(\alpha, \beta, s) = \sum_{m,n} m^{\alpha+\beta-s} n^{-s} \sum_{\substack{h \le y_1 \\ k \le y_4}} \frac{\mathfrak{b}(h)\mathfrak{c}(k)}{h^{1-s+\beta} k^{1-s+\alpha}} e\left(\frac{mn\bar{H}}{K}\right), \tag{5.59}$$

where $H = h/(h,k)$, $K = k/(h,k)$ and $e(x) = e^{2\pi i x}$. The fact that stops us from following the next step in [Con89] is that $\mathfrak{c}(k) \neq \mu(k)F(k)$, for some smooth function F. We estimate (5.59) trivially. Using the fact $\mathfrak{b}(h) \ll h^{\varepsilon}$ and $\mathfrak{c}(k) \ll k^{\varepsilon}$ we have

$$\mathcal{M}(\alpha, \beta, s) \ll y_1^{1+\varepsilon+\eta} y_4^{1+\varepsilon+\eta}$$

for $s = 1 + \eta + it$ and $\eta \ll 1/L$. This gives us the required error term. □

Following the same ideas, we also have

Proposition 5.20. *Let $\theta_4 < 1/2 - \varepsilon$ and $T/2 \le w \le T$. Then we have*

$$I_{44}(\alpha, \beta, w) = \frac{\sum'(\beta, \alpha) - e^{-(\alpha+\beta)L} \sum'(-\alpha, -\beta)}{\alpha + \beta} + O(L^{-1+\varepsilon}), \tag{5.60}$$

where

$$\sum{}'(\beta, \alpha) = \sum_{n,m \le y_4} \frac{\mathfrak{c}(n)\mathfrak{c}(m)}{n^{1+\alpha} m^{1+\beta}} (n, m)^{1+\alpha+\beta}. \tag{5.61}$$

Combining the main term of $I_{11}(\alpha, \beta, w)$, $I_{14}(\alpha, \beta, w)$, and $I_{44}(\alpha, \beta, w)$ yields the main term of Lemma 2 of [Fen12] provided that $\theta_1 < 4/7 - \varepsilon$ and $\theta_4 < 3/7 - \varepsilon$. This completes the proof of Proposition 5.7.

5.8 Proof of Proposition 5.8

When we insert the definitions of the mollifiers

$$\psi_1(s) = \sum_{a \le y_1} \frac{\mu(a)}{a^{s-\sigma_0-1/2}} P_1[a],$$

and

$$\psi_3(s) = \chi^2(s + \tfrac{1}{2} - \sigma_0) \sum_{bc \le y_3} \frac{\mu_3(b)d(c)}{b^{s-\sigma_0-1/2} c^{1/2-s+\sigma_0}} P_3[bc],$$

in the mean-value integral we have

$$\begin{aligned} I_{13}(\alpha, \beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2} + \alpha + it)\zeta(\tfrac{1}{2} + \beta - it)\overline{\psi_1}\psi_3(\sigma_0 + it)dt \\ &= \sum_{a \le y_1} \sum_{bc \le y_3} \frac{\mu(a)\mu_3(b)d(c)}{(abc)^{1/2}} P_1[a]P_3[bc]J_{13}, \end{aligned} \tag{5.62}$$

where

$$J_{13} = \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\chi^2(\tfrac{1}{2}+it)\left(\frac{b}{ac}\right)^{-it} dt.$$

Using the same procedure as in the previous section (i.e. approximation of $\chi(\frac{1}{2}+\beta-it)\chi(\frac{1}{2}+it)$, followed by the functional equation of $\zeta(\frac{1}{2}+\beta-it)$), we obtain

$$J_{13} = \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)\chi(\tfrac{1}{2}+it)dt + O(T^{\varepsilon}).$$

From the Stirling formula we have

$$\chi(\tfrac{1}{2}+it) = F(t) + E(T),$$

where

$$F(t) = e^{i\pi/4}\left(\frac{t}{2\pi e}\right)^{-it} \quad \text{and} \quad E(t) \ll \frac{1}{T}.$$

Note that

$$\begin{aligned}&\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)E(t)dt \\ &\ll \frac{1}{T}\int_{T/4}^{2T} |\zeta(\tfrac{1}{2}+\alpha+it)||\zeta(\tfrac{1}{2}-\beta+it)|dt \ll \frac{1}{T}\log T.\end{aligned}$$

Thus, we are left with

$$J_{13} = \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}-\beta+it)F(t)dt + O_{\varepsilon}(T^{\varepsilon}).$$

Now we use Lemma 5.10 to see that

$$\begin{aligned}J_{13} &= \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\left(\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3} + O(T^{-1+\varepsilon})\right)F(t)dt + O_{\varepsilon}(T^{\varepsilon}) \\ &= \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{b}{ac}\right)^{-it}\left(\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}\right)F(t)dt + O_{\varepsilon}(T^{\varepsilon}) \\ &= \sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3}\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{bl}{ac}\right)^{-it}F(t)dt + O_{\varepsilon}(T^{\varepsilon}).\end{aligned}$$

For all $1 \le a \le y_1$, $1 \le b \le y_3$, $1 \le c \le y_3$ and any $l \ge 1$ we have

$$\frac{tbl}{2\pi eac} \ge \frac{T}{4}\frac{1}{2\pi e y_1 y_3} = \frac{T}{8\pi e T^{\theta_1+\theta_3}} \ge T^{\varepsilon_0}, \tag{5.63}$$

provided $\theta_1 < 4/7 - \varepsilon$ and $\theta_3 < 3/7 - \varepsilon$. We also recall the fact $w^{(r)}(t) \ll (L/T)^r$. Therefore from (5.63) and by the aid of integration by parts we have

$$\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta}\left(\frac{tbl}{2\pi eac}\right)^{-it} dt \ll_{r,\varepsilon_0} \frac{1}{T^r}$$

for any fixed integer r. This leaves us with

$$J_{13} \ll_{r,\varepsilon_0} \frac{1}{T^r}\sum_{l=1}^{\infty}\frac{\sigma_{\alpha,-\beta}(l)}{l^{1/2+it}}e^{-l/T^3} + O_\varepsilon(T^\varepsilon) \ll_{\varepsilon_0,\varepsilon} T^\varepsilon.$$

Putting this back into $I_{13}(\alpha,\beta)$ we see that

$$I_{13} \ll_{\varepsilon_0,\varepsilon} T^\varepsilon \sum_{\substack{a\le y_1\\ bc\le y_3}}\frac{|\mu(a)\mu_3(b)d(c)|}{(abc)^{1/2}}|P_1[a]P_3[bc]| \ll_{\varepsilon_0,\varepsilon} T^{2\varepsilon}\sum_{\substack{a\le y_1\\ bc\le y_3}}\frac{1}{(abc)^{1/2}},$$

since P_1 and P_3 are real polynomials in logarithms. Finally, we have

$$\begin{aligned} I_{13} &\ll_{\varepsilon_0,\varepsilon} T^{2\varepsilon}\left(\sum_{a\le y_1}\frac{1}{\sqrt{a}}\right)\left(\sum_{m\le y_3}\frac{d(m)}{\sqrt{m}}\right) \ll_{\varepsilon_0,\varepsilon} T^{3\varepsilon}y_1^{1/2}y_3^{1/2} \ll_{\varepsilon_0,\varepsilon} T^{3\varepsilon+(\theta_1+\theta_3)/2} \\ &= T^{\frac{1}{2}+3\varepsilon-2\varepsilon_0}. \end{aligned} \tag{5.64}$$

This completes the proof the proposition.

5.9 Proof of Proposition 5.9

First we note that the extra term of the logarithms satisfies

$$\frac{\log p_1\cdots\log p_k}{\log^k y_4} \ll 1.$$

Moreover, their sum is

$$\sum_{p_1\cdots p_k|c}\frac{\log p_1\cdots\log p_k}{\log^k y_4}\widetilde{P}_k[c] \ll d(c) \ll c^\varepsilon.$$

Hence, this proof follows the exact same procedure as when we dealt with the cross term $I_{13}(\alpha,\beta)$ in Section 5.8.

5.10 Application

Let $N(T)$ denote the number of zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ with $0<\gamma<T$ and $0<\beta<1$. Let $N_0(T)$ denote the number of such zeros with $\beta=\frac{1}{2}$, and let $N_0^*(T)$ denote the number of such zeros and which are simple as well. We define κ and κ^* by

$$\kappa = \liminf_{T\to\infty}\frac{N_0(T)}{N(T)}, \quad \kappa^* = \liminf_{T\to\infty}\frac{N_0^*(T)}{N(T)}.$$

In 1942 Selberg [Sel42] proved that $\kappa>0$; in other words, a positive proportion of the zeros of the Riemann zeta-function lies on the critical line $\sigma=\frac{1}{2}$. Since then there have been improvements on the actual value of κ. Of these results we note Levinson's 1974 [Lev74] result that $\kappa\ge .3474$. In 1985, Conrey and Ghosh [CG85] simplified Levinson's method and later in 2010, Young [You10] gave a much shorter proof of Levinson's result.

In 1989, Conrey [Con89] used deep arithmetical results on Kloosterman sums due to Deshouillers and Iwaniec [DI82, DI84] and his own analytic devices [BCHB85, Con83a, Con83b] to set the record at $\kappa \geq .4088$. In the early 2010's Bui, Conrey and Young [BCY11], and slightly afterward Feng [Fen12], improved this to $\kappa \geq .4105$ and $\kappa \geq .4128$, respectively. However, as mentioned in the introduction, the result $\kappa \geq .4128$ is not clear. In this section we provide the following application of Theorem 5.1.

Theorem 5.21. *We have*

$$\kappa \geq .410725 \quad \text{and} \quad \kappa^* \geq .405824.$$

Let $Q(x)$ be a real polynomial satisfying $Q(0) = 1$ as well as $Q(x)+Q(1-x) = \text{constant}$, and define

$$V(s) = Q\left(-\frac{1}{L}\frac{d}{ds}\right)\zeta(s).$$

Since

$$|\psi_2(s)| \ll \sqrt{t}\Big(\frac{y_2}{t}\Big)^{\sigma} L^2 \quad \text{and} \quad |\psi_3(s)| \ll t\Big(\frac{y_3}{t^2}\Big)^{\sigma} L^4, \tag{5.65}$$

then $\log\psi(s)$ is analytic. Hence $\psi(s)$ is a valid mollifier in Levinson's method (see [Lev74]) and it satisfies the inequality

$$\kappa \geq 1 - \frac{1}{R}\log\left(\frac{1}{T}\int_1^T |V\psi(\sigma_0+it)|^2 dt\right) + o(1),$$

where $\sigma_0 = 1/2 - R/L$, and where R is a bounded positive real number of our choice. Choosing $Q(x)$ to be a linear polynomial yields a lower bound on the percent of simple zeros κ^*. Let us denote the integral in (5.9) by $I(\alpha,\beta)$. Then we have

$$\int_1^T |V\psi(\sigma_0+it)|^2 dt = Q\left(\frac{-1}{L}\frac{d}{d\alpha}\right) Q\left(\frac{-1}{L}\frac{d}{d\beta}\right) I(\alpha,\beta)\Big|_{\alpha=\beta=-R/L}. \tag{5.66}$$

Also one has

$$Q\left(\frac{-1}{\log T}\frac{d}{d\alpha}\right) X^{-\alpha} = Q\left(\frac{\log X}{\log T}\right) X^{-\alpha}. \tag{5.67}$$

Combining (5.66) and (5.67) we have

Theorem 5.22. *Suppose that $\theta_1 = 4/7-\varepsilon$, $\theta_2 = 1/2-\varepsilon$, $\theta_3 = 3/7-\varepsilon$ and $\theta_4 = 3/7-\varepsilon$ for $\varepsilon > 0$ small. Then*

$$\int_1^T |V\psi(\sigma_0+it)|^2 dt = cT + O_\varepsilon(TL^{-1+\varepsilon}),$$

where c is an explicit constant that depends on $Q, P_1, P_2, P_3, R, \theta_1, \theta_2, \theta_3, \theta_4$ and $\tilde{P}_k$ for $k = 2, 3, \ldots, K$.

The constant c is given by $c = c_{11} + 2c_{12} + c_{22} + 2c_{23} + c_{33} + 2c_{14} + 2c_{24} + c_{44}$. The value of $c_{11} + 2c_{14} + c_{44}$ was given in the main term of [Fen12, Eq. (5.3)]. The expressions of c_{12} and c_{22} were given in [BCY11, Eq. (3.4) and Eq. (3.6)]. The remaining values, i.e. c_{23}, c_{33}, c_{13}, c_{24} and c_{34} are now given below. Applying (5.67) on Propositions 5.4 and

5.5 and setting $\alpha = \beta = -R/L$, we get

$$\begin{aligned} c_{23} = \frac{2^8}{7!}\left(\frac{\theta_3}{\theta_2}\right)^6 e^R \frac{d^6}{dx^3dy^3}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}} \int_0^1 & u^4(1-u)^7 e^{R[\theta_2(y-x)+u\theta_3(a-b)]} Q(-x\theta_2 + au\theta_3) \\ & \times Q(1+y\theta_2 - bu\theta_3) P_2''\left(x+y+1-(1-u)\frac{\theta_3}{\theta_2}\right) \\ & \times abP_3^{(6)}((1-a-b)u)dudadb\Bigg]_{x=y=0}, \end{aligned} \tag{5.68}$$

and

$$\begin{aligned} c_{33} = \frac{2^{12}}{12!}\frac{d^6}{dx^3dy^3}\Bigg(\int_0^1\int_0^1\int_0^1\int_0^1 & \left(\frac{1}{\theta_3} + (x+y-v(y+r)-u(x+r))\right) \\ & \times (1-r)^{12} e^{-\theta_3 R(x+y-v(y+r)-u(x+r))} e^{2Rt(1+\theta_3(x+y-v(y+r)-u(x+r)))} \\ & \times Q(\theta_3(-x+v(y+r)) + t(1+\theta_3(x+y-v(y+r)-u(x+r)))) \\ & \times Q(\theta_3(-y+u(x+r)) + t(1+\theta_3(x+y-v(y+r)-u(x+r)))) \\ & \times ((1-u)(x+r))P_3^{(6)}((1-v)(y+r)) \\ & \times (x+r)^2(y+r)^2P_3^{(6)} dtdrdudv\Bigg)\Bigg|_{x=y=0}. \end{aligned}$$

Finally, from using (5.67) on Proposition 5.6 and setting $\alpha = \beta = -R/L$, we obtain

$$\begin{aligned} c_{24} = c_{42} = 4e^R\sum_{k=2}^{K}(& c_{42}^{(0,0)}(k) + c_{42}^{(0,1)}(k) + c_{42}^{(1,0)}(k) + c_{42}^{(1,1)}(k) + c_{42}^{(1,\ge 2)}(k) + c_{42}^{(\ge 2,1)}(k) \\ & + c_{42}^{(0,\ge 2)}(k) + c_{42}^{(\ge 2,0)}(k) + c_{42}^{(\ge 2,\ge 2)}(k)), \end{aligned}$$

where

$$\begin{aligned} c_{42}^{(0,0)}(k) = & \frac{2^k}{(k+1)!}\frac{d^2}{dxdy}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 (1-u)^{1+k} \\ & \times e^{R[\theta_2(a-x)+\theta_4 a(u-1)]+R[\theta_2(y-b)+\theta_4 b(-u+1)]} \\ & \times \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 \widetilde{P}_k(x+y+u)P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right) \\ & \times Q(\theta_2(a-x)+\theta_4 a(u-1))Q(\theta_2(y-b)+\theta_4 b(-u+1)+1)dudadb\Bigg]_{x=y=0}, \end{aligned}$$

$$\begin{aligned} c_{42}^{(1,0)}(k) = -\frac{2^{k-1}}{(k-1)!}\frac{d}{dy}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 & (1-u)^k e^{R[-\theta_4(1-u)a+\theta_2 a]+R[\theta_4(b(1-u)+y)-\theta_2 b]} \\ & \times \left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(y+u) \\ & \times Q(-\theta_4(1-u)a+\theta_2 a)Q(\theta_4(b(1-u)+y)-\theta_2 b+1)dudadb\Bigg]_{y=0}, \end{aligned}$$

$$
\begin{aligned}
c_{42}^{(0,1)}(k) = -\frac{2^{k-1}}{(k-1)!}\frac{d}{dx}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 &(1-u)^k e^{R[-\theta_4((1-u)a+x)+\theta_2 a]}e^{R[\theta_4 b(1-u)-\theta_2 b]}\\
&\times\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(x+u)\\
&\times Q(-\theta_4((1-u)a+x)+\theta_2 a)Q(\theta_4 b(1-u)-\theta_2 b+1)dudadb\Bigg]_{x=0},
\end{aligned}
$$

$$
\begin{aligned}
c_{42}^{(1,1)}(k) = \frac{2^{k-2}k}{(k-2)!}\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1 &(1-u)^{k-1}e^{R[\theta_4(-(1-u)a)+\theta_2 a]}e^{R[\theta_F b(1-u)-\theta_2 b]}\\
&\times\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k(u)\\
&\times Q(\theta_4(-(1-u)a)+\theta_2 a)Q(\theta_F b(1-u)-\theta_2 b+1)dudadb,
\end{aligned}
$$

$$
\begin{aligned}
c_{42}^{(1,\ge 2)}(k) = -k!\sum_{l_1+1+l_3=k}&\frac{2^{l_1}(-1)^{l_3-2}}{l_1!l_3!(1+l_1)!(l_3-2)!}\\
&\times\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2(1-u)^{1+l_1}\\
&\times e^{R[\theta_4 a(u-1)+\theta_2 a]}e^{R[\theta_4(b(1-u)-uc)-\theta_2 b]}\\
&\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k((1-c)u)u^{l_3-1}c^{l_3-2}\\
&\times Q(\theta_4 a(u-1)+\theta_2 a)Q(\theta_4(b(1-u)-uc)-\theta_2 b+1)dudcdadb,
\end{aligned}
$$

with $l_3\ge 2$,

$$
\begin{aligned}
c_{42}^{(\ge 2,1)} = -k!\sum_{l_1+l_2+1=k}&\frac{2^{l_1}(-1)^{l_2-2}}{l_1!l_2!(1+l_1)!(l_2-2)!}\\
&\times\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\left(1-(1-u)\frac{\theta_4}{\theta_4}\right)^2(1-u)^{1+l_1}\\
&\times e^{R[\theta_4(a(u-1)+uc)+\theta_2 a]}e^{R[\theta_4 b(1-u)-\theta_2 b]}\\
&\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_4}\right)\right)\widetilde{P}_k((1-c)u)u^{l_2-1}c^{l_2-2}\\
&\times Q(\theta_4(a(u-1)+uc)+\theta_2 a)Q(\theta_4 b(1-u)-\theta_2 b+1)dudcdadb,
\end{aligned}
$$

with $l_2\ge 2$,

$$
\begin{aligned}
c_{42}^{(\ge 2,0)} = k!\sum_{l_1+l_2=k}&\frac{2^{l_1}(-1)^{l_2}}{l_1!l_2!(l_2-2)!(1+l_1)!}\frac{d}{dx}\Bigg[\iint_{\substack{0\le a+b\le 1\\ a,b\ge 0}}\int_0^1\int_0^1\\
&\times(1-u)^{1+l_1}(x+u)^{l_2-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_2-2}\\
&\times e^{R[\theta_4(c(u+x)-(1-u)a)+\theta_2 a]+R[\theta_4(x+(1-u)b)-\theta_2 b]}\\
&\times Q(\theta_4(c(u+x)-(1-u)a)+\theta_2 a)Q(\theta_4(x+(1-u)b)-\theta_2 b+1)
\end{aligned}
$$

$$\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(x+u))dadbdcdu\Bigg]_{x=0}$$

with $l_2 \geq 2$,

$$\begin{aligned}
c_{42}^{(0,\geq 2)} &= k! \sum_{l_1+l_3=k} \frac{2^{l_1}(-1)^{l_3}}{l_1!l_3!(l_3-2)!(1+l_1)!}\frac{d}{dy}\Bigg[\iint_{\substack{0\leq a+b\leq 1\\ a,b\geq 0}}\int_0^1\int_0^1 \\
&\times (1-u)^{1+l_1}(y+u)^{l_3-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 c^{l_3-2} \\
&\times e^{R[\theta_4(-y-a(1-u))+\theta_2 a]+R[\theta_4(-c(u+y)+b(1-u))-\theta_2 b]} \\
&\times Q(\theta_4(-y-a(1-u))+\theta_2 a)Q(1-\theta_2 b+\theta_4(-c(u+y)+b(1-u))) \\
&\times P''_2\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\tilde{P}_k((1-c)(y+u))dadbdcdu\Bigg]_{y=0}
\end{aligned}$$

with $l_3 \geq 2$,

$$\begin{aligned}
c_{42}^{(\geq 2,\geq 2)}(k) = k! &\sum_{l_1+l_2+l_3=k} \frac{2^{l_1}(-1)^{l_2+l_3}}{l_1!l_2!l_3!(1+l_1)!(l_2-2)!(l_3-2)!} \\
&\times \iiiint_{\substack{0\leq a+b\leq 1\\ 0\leq g+h\leq 1\\ a,b,g,h\geq 0}}\int_0^1 (1-u)^{k+l-1}\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)^2 \\
&\times e^{R[\theta_4(au+gu-a)+\theta_2 a]}e^{R[\theta_4(b-bu-hu)-\theta_2 b]} \\
&\times P_2''\left((1-a-b)\left(1-(1-u)\frac{\theta_4}{\theta_2}\right)\right)\widetilde{P}_k((1-g-h)u) \\
&\times Q(\theta_4(au+gu-a)+\theta_2 a)Q(\theta_4(b-bu-hu)-\theta_2 b+1) \\
&\times u^{l_2+l_3-2}g^{l_2-2}h^{l_3-2}dudadbdgdh,
\end{aligned}$$

with $l_2 \geq 2$ and $l_3 \geq 2$.

Finally, we use `Mathematica` to numerically evaluate c with the following particular choices of parameters. Set $R = 1.295$, $\theta_1 = 4/7$, $\theta_2 = 1/2$, $\theta_3 = 3/7$, $\theta_4 = 3/7$ and $K = 3$,

$$\begin{aligned}
Q(x) &= 0.492203 + 0.621972(1-2x) - 0.148163(1-2x)^3 + 0.033988(1-2x)^5 \\
P_1(x) &= x + 0.229117x(1-x) - 2.932318x(1-x^2) + 4.856163x(1-x^3) \\
&\quad - 2.309993x(1-x^4), \\
\tilde{P}_2(x) &= -0.072644x + 1.559440x^2, \\
\tilde{P}_3(x) &= 0.701568x - 0.554403x^2,
\end{aligned}$$

and keep the coefficients of all the other polynomials temporarily set to zero, we then have $\kappa \geq .410725$.

Moreover, by setting $R = 1.1195$, $\theta_1 = 4/7$, $\theta_2 = 1/2$, $\theta_3 = 3/7$, $\theta_4 = 3/7$ and taking,

$$\begin{aligned}
Q^*(x) &= .483872 + .516128(1-2x), \\
P_1^*(x) &= .827329x + .0108498x^2 + .0815758x^3 + .181027x^4 - .100781x^5, \\
P_2^*(x) &= .0326349x^3 - .0056269x^4 + .00783646x^5,
\end{aligned}$$

and keeping the coefficients of all the other polynomials temporarily set to zero, we then get $\kappa^* \geq .405824$.

Chapter 6

Zeta functions on tori using contour integration

6.1 Introduction

Zeta regularization and the theory of spectral zeta functions are powerful and elegant techniques that allow one to assign finite values to otherwise manifestly infinite quantities in a unique and well-defined way [Eli12, EOR+94, Kir02].
Suppose we have a compact smooth manifold M with a Riemannian metric g and a corresponding Laplace-Beltrami operator $\Delta = \Delta(g)$, where Δ has a discrete spectrum

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{j \to \infty} \lambda_j = \infty. \tag{6.1}$$

If we denote by n_j the finite multiplicity of the j-th eigenvalue λ_j of Δ then, by a result of H. Weyl [Wey12], which says that the asymptotic behavior of the eigenvalues as $j \to \infty$ is $\lambda_j \sim j^{2/\dim M}$, we can construct the corresponding spectral zeta function as

$$\zeta_M(s) = \sum_{j=1}^{\infty} \frac{n_j}{\lambda_j^s}, \tag{6.2}$$

which is well-defined for $\mathrm{Re}(s) > \frac{1}{2}\dim M$. Minakshisundaram and Pleijel [MP49] showed that $\zeta_M(s)$ admits a meromorphic continuation to the whole complex plane and that, in particular, $\zeta_M(s)$ is holomorphic at $s = 0$. This, in turn, means that $\exp[-\zeta_M'(0)]$ is well-defined and Ray and Singer [RS71] set the definition

$$\det(\Delta) = \prod_{k=1}^{\infty} \lambda_k^{n_k} := e^{-\zeta_M'(0)}, \tag{6.3}$$

where it is understood that the zero eigenvalue of Δ is not taken into the product. The motivation for this definition comes from the formal computation

$$\exp\left[-\frac{d}{ds}\bigg|_{s=0} \sum_{k=1}^{\infty} \frac{n_k}{\lambda_k^s}\right] = \exp\left[\sum_{k=1}^{\infty} n_k \log \lambda_k\right] = \prod_{k=1}^{\infty} e^{n_k \log \lambda_k} = \prod_{k=1}^{\infty} \lambda_k^{n_k}. \tag{6.4}$$

As long as the spectrum is discrete, the definition coming from (6.3) is suitable for more general operators on other infinite dimensional spaces. In particular, it is useful for Laplace-type operators on smooth manifolds of a vector bundle over M, see e.g. [Nak03, Wil10].

6.2 Argument principle technique

Let us introduce the basic ideas used in this chapter by considering a generic one dimensional second order differential operator $\mathcal{O} := -d^2/dx^2 + V(x)$ on the interval $[0, 1]$, where $V(x)$ is a smooth potential. Let its eigenvalue problem be given by

$$\mathcal{O}\phi_n(x) = \lambda_n \phi_n(x), \tag{6.5}$$

and choose Dirichlet boundary conditions $\phi_n(0) = \phi_n(1) = 0$. This problem can be translated to a unique initial value problem [KL08, Kir10, LS75]

$$(\mathcal{O} - \lambda)u_\lambda(x) = 0, \tag{6.6}$$

with $u_\lambda(0) = 0$ and $u'_\lambda(0) = 1$. The eigenvalues λ_n then follow as the solutions to the equation

$$u_\lambda(1) = 0, \tag{6.7}$$

where $u_\lambda(1)$ is an analytic function of λ. Let us recall the argument principle from complex analysis. It states that if f is a meromorphic function inside and on some counterclockwise contour γ with f having neither zeroes nor poles on γ, then

$$\int_\gamma dz\, \frac{f'(z)}{f(z)} = 2\pi i(N - P), \tag{6.8}$$

where N and P are, respectively, the number of zeros and poles of f inside the contour γ. There is a slightly stronger version of this statement called the generalized argument principle, stating that if f is meromorphic in a simply connected set D which has zeroes a_j and poles b_k, if g is an analytic function in D, and if we let γ be a closed curve in D avoiding a_j and b_k, then

$$\sum_j g(a_j)n(\gamma, a_j) - \sum_k g(b_k)n(\gamma, b_k) = \frac{1}{2\pi i}\int_\gamma dz\ g(z)\frac{f'(z)}{f(z)}, \tag{6.9}$$

where $n(\gamma, a)$ is the winding number of the closed curve γ with respect to the point $a \notin \gamma$, defined as

$$n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - a} \in \mathbb{Z}. \tag{6.10}$$

If we let f be a polynomial with zeros $z_1, z_2, \cdots$ and $g(z) := z^s$, then

$$\frac{1}{2\pi i}\int_\gamma dz\ z^s\frac{f'(z)}{f(z)} = z_1^s + z_2^s + \cdots\ , \tag{6.11}$$

or equivalently

$$\frac{1}{2\pi i}\int_\gamma dz\ z^s \frac{d}{dz}\log f(z) = \sum_n z_n^s. \tag{6.12}$$

Taking into account the asymptotic properties of $u_\lambda(1)$ and making the substitutions $z \to \lambda$ and $s \to -s$, we see that [Kir02, KL08, Kir10]

$$\frac{1}{2\pi i}\int_\gamma d\lambda\lambda^{-s} \frac{d}{d\lambda}\log u_\lambda(1) = \sum_n \lambda_n^{-s} := \zeta_{\mathcal{O}}(s), \tag{6.13}$$

since the eigenvalues λ_n are solutions of $u_\lambda(1) = 0$. As before, γ is a counterclockwise contour that encloses all eigenvalues, which we assume to be positive; see Fig. 6.1. The

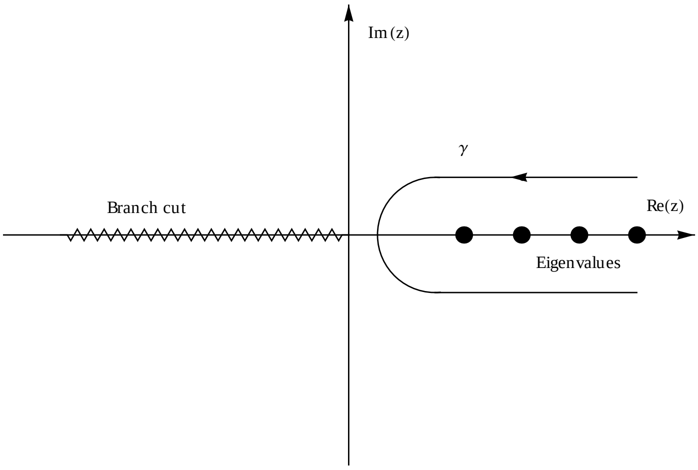

FIGURE 6.1: Integration contour γ.

pertinent remarks for the case when finitely many eigenvalues are non-positive are given in [KM04]. It is important to note that the asymptotic behavior of $u_\lambda(1)$ as $|\lambda| \to \infty$ is given by [Kir10, LS75]

$$u_\lambda(1) \sim \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}}. \tag{6.14}$$

This implies that the integral representation for $\zeta_{\mathcal{O}}(s)$ is valid for $\mathrm{Re}(s) > \frac{1}{2}$ and, therefore, we must continue it analytically if we are to take its derivative at $s = 0$ to compute the determinant of $\mathcal{O}$.

The next step [KL08, Kir10] necessary to evaluate this integral is to deform the contour suitably. These deformations are allowed provided one does not cross over poles or branch cuts of the integrand. By assumption, for our integrand the poles are on the real axis and, as customary, we define the branch cut of λ^{-s} to be on the negative real axis. This means that, as long as the behavior at infinity is appropriate, we are allowed to deform the contour to the one given in Fig. 6.2. The result of this deformation, after shrinking the contour to the negative real axis, is

$$\zeta_{\mathcal{O}}(s) = \frac{\sin \pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{d}{d\lambda}\log u_{-\lambda}(1). \tag{6.15}$$

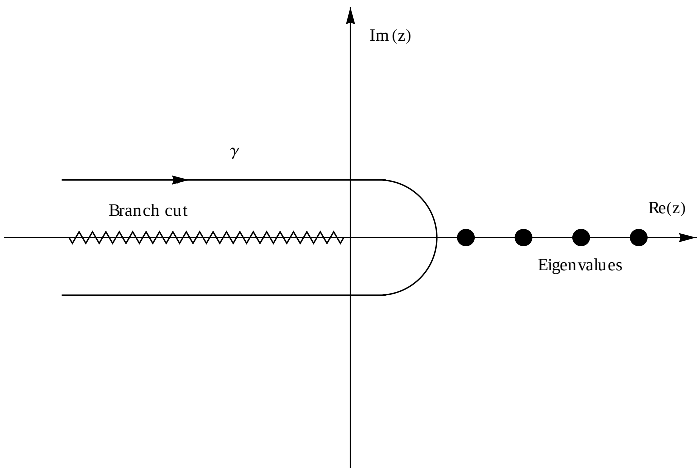

FIGURE 6.2: Deformed integration contour γ.

To establish the limits of the validity of this integral representation we must examine the behavior of the integrand for $\lambda \to \infty$, namely one has [KM04, Kir10]

$$u_{-\lambda}(1) \sim \frac{\sin(i\sqrt{\lambda})}{(i\sqrt{\lambda})} \sim \frac{e^{\sqrt{\lambda}}}{2\sqrt{\lambda}}. \tag{6.16}$$

Thus, to leading order in λ, the integrand behaves as $\lambda^{-s-\frac{1}{2}}$, which means that convergence at infinity is established for $\mathrm{Re}(s) > \frac{1}{2}$, as we discussed above. On the other hand, when $\lambda \to 0$ the behavior λ^{-s} follows. Consequently the integral representation (6.15) is well-defined for

$$\frac{1}{2} < \mathrm{Re}(s) < 1. \tag{6.17}$$

The analytic continuation to the left is accomplished by subtracting the leading $\lambda \to \infty$ asymptotic behavior of $u_{-\lambda}(1)$ [Kir02]. Carrying out this procedure results in one part that is finite at $s = 0$ and another part for which the analytic continuation can be constructed relatively easily.
The partition of $\zeta_{\mathcal{O}}$ that keeps the $\lambda \to 0$ term unaffected and that should improve the $\lambda \to \infty$ part is accomplished by splitting the integration range as

$$\zeta_{\mathcal{O}}(s) = \zeta_{\mathcal{O},\mathrm{asy}}(s) + \zeta_{\mathcal{O},\mathrm{f}}(s), \tag{6.18}$$

where

$$\zeta_{\mathcal{O},\mathrm{f}}(s) = \frac{\sin \pi s}{\pi} \int_0^1 d\lambda \lambda^{-s} \frac{d}{d\lambda} \log u_{-\lambda}(1) + \frac{\sin \pi s}{\pi} \int_1^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \log \left[u_{-\lambda}(1) \left(\frac{2\sqrt{\lambda}}{e^{\sqrt{\lambda}}} \right) \right], \tag{6.19}$$

and

$$\zeta_{\mathcal{O},\mathrm{asy}}(s) = \frac{\sin \pi s}{\pi} \int_1^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \log \frac{e^{\sqrt{\lambda}}}{2\sqrt{\lambda}}. \tag{6.20}$$

Clearly, we have constructed $\zeta_{\mathcal{O},\mathrm{f}}(s)$ in such a way that it is already analytic at $s = 0$ and, thus, its derivative at $s = 0$ can be computed immediately

$$\zeta'_{\mathcal{O},\mathrm{f}}(0) = -\log[2e^{-1}u_0(1)]. \tag{6.21}$$

For this case, the analytic continuation to a meromorphic function on the complex plane now follows from

$$\int_1^\infty d\lambda\lambda^{-\alpha} = \frac{1}{\alpha - 1} \quad \text{for} \quad \mathrm{Re}(\alpha) > 1. \tag{6.22}$$

Applying the above to $\zeta_{\mathcal{O},\mathrm{asy}}(s)$ yields

$$\zeta_{\mathcal{O},\mathrm{asy}}(s) = \frac{\sin \pi s}{2\pi}\left(\frac{1}{s - 1/2} - \frac{1}{s}\right), \tag{6.23}$$

and thus

$$\zeta'_{\mathcal{O},\mathrm{asy}}(0) = -1. \tag{6.24}$$

The contribution from both terms then becomes

$$\zeta'_{\mathcal{O}}(0) = -\log[2u_0(1)]. \tag{6.25}$$

Note how we could numerically evaluate the determinant of $\mathcal{O}$ without using a single eigenvalue explicitly [Kir10].

6.3 Description of the problem

Let us next introduce the notions needed for the investigations of the Eisenstein series. Let M be a compact smooth manifold with dimension d and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > d/2$, furthermore let $\mathbb{H}$ denote the upper half-plane $\mathbb{H} = \{\tau = \tau_1 + i\tau_2, \tau_1 \in \mathbb{R}, \tau_2 > 0\}$.

Definition 6.1. For $c \in \mathbb{R}_+$ and $\vec{r} \in \mathbb{R}_+^d$ the homogeneous Epstein zeta function is defined as [Eps03, Eps07]

$$\zeta_{\mathcal{E}}(s, c|\vec{r}) := \sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{(c + r_1 m_1^2 + \cdots + r_d m_d^2)^s}. \tag{6.26}$$

If $c = 0$ then it is understood that the summation ranges over $\vec{m} \neq \vec{0}$.

Definition 6.2. Let $\mathbb{Z}_*^2 = \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and $\tau \in \mathbb{H}$ with $\mathrm{Re}\,\tau = \tau_1$ and $\mathrm{Im}\,\tau = \tau_2$. For $\mathrm{Re}(s) > 1$ the nonholomorphic Eisenstein series is defined as [Wil10]

$$E^*(s, \tau) := \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{\tau_2^s}{|m + n\tau|^{2s}}. \tag{6.27}$$

Note that for $\tau = i$ the nonholomorphic Eisenstein series is related to the homogeneous Epstein zeta function by

$$E^*(s, i) = \sum_{(m,n)\in\mathbb{Z}_*^2} \frac{1}{(m^2 + n^2)^s} = \zeta_{\mathcal{E}}(s, 0|\vec{1}_2), \tag{6.28}$$

where $\vec{1}_2 = (1,1)$. The non-holomorphic Eisenstein series is not holomorphic in τ but it can be continued analytically beyond $\mathrm{Re}(s) > 1$ except at $s = 1$, where there is a simple pole with residue equal to π.

Definition 6.3. For $\tau \in \mathbb{H}$, the Dedekind eta function is defined as

$$\eta(\tau) := e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}).$$

While $\eta(\tau)$ is holomorphic on the upper half-plane, it cannot be continued analytically beyond it. The fundamental properties are that it satisfies the following functional equations.

Proposition 6.4. *One has*

$$\begin{aligned}\eta(\tau+1) &= e^{\pi i/12}\eta(\tau),\\ \eta(-\tau^{-1}) &= \sqrt{-i\tau}\,\eta(\tau),\end{aligned}$$

for $\tau \in \mathbb{H}$.

The first equation is very easy to show and the proof of the second one can be found in most books on modular forms, see for instance [Bum98]. With this in mind, the constant term in the Laurent expansion of $E^*(s,\tau)$ is given in the following theorem (see, e.g., [Wil10]).

Theorem 6.5 (Kronecker's first limit formula)**.**

$$\lim_{s\to 1}\left(E^*(s,\tau) - \frac{\pi}{s-1}\right) = 2\pi(\gamma - \log 2 - \log \tau_2^{1/2}\,|\eta(\tau)|^2).$$

A modular form of weight $k > 0$ and multiplier condition C for the group of substitutions generated by $\tau \to \tau + \lambda$ and $\tau \to -\frac{1}{\tau}$ is a holomorphic function $f(\tau)$ on $\mathbb{H}$ satisfying [GM03]

(i) $f(\tau + \lambda) = f(\tau)$,

(ii) $f(-\frac{1}{\tau}) = C(\frac{\tau}{i})^k f(\tau)$,

(iii) $f(\tau)$ has a Taylor expansion in $e^{(2\pi i\tau/\lambda)}$ (cf (i)): $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda}$, i.e. "$f$ is holomorphic at ∞".

The space of such f is denoted by $M(\lambda, k, C)$ and furthermore if $a_0 = 0$ then f is a cusp form. The group of substitutions generated by $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ is

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ such that } a,b,c,d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\},$$

therefore modular forms of weight k satisfy

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

The Dedekind eta function is a modular form of weight $k = \frac{1}{2}$. Moreover, it satifies

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon(a,b,c,d)\{-i(c\tau+d)\}^{1/2}\eta(\tau),$$

where $\varepsilon(a,b,c,d) = e^{b\pi i/12}$ if $c = 0$, and if $c > 0$ then

$$\varepsilon(a,b,c,d) = \exp\left(i\pi\left\{\frac{a+d}{12c} - s(-d,c)\right\}\right),$$

where $s(h,k)$ is the Dedekind sum

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right).$$

Finally, the non-holomorphic Eisenstein series can alternatively be defined as

$$E^*(s,\tau) = \zeta_R(2s) \sum_{\substack{(m,n)\\ \gcd(m,n)=1}} \frac{\tau_2^s}{|m\tau+n|^{2s}},$$

where ζ_R(s) is the Riemann zeta function, and it is unchanged by the substitutions

$$\tau \to \frac{a\tau+b}{c\tau+d}$$

coming from any matrix of $\mathrm{SL}(2,\mathbb{Z})$. Selberg and Chowla were interested in the problem of the analytic continuation of $E^*(s,\tau)$ as a function of s and its functional equation. Their idea was to consider the Fourier expansion of $E^*(s,\tau)$ given by

$$E^*(s,\tau) = E(s,\tau_1+i\tau_2) = \sum_{m\in\mathbb{Z}} a_m(s,\tau_2)e^{2\pi i m\tau_1},$$

where a_m is the Fourier coefficient

$$a_m(\tau_2,s) = \int_0^1 E(s,\tau_1+i\tau_2)e^{-2\pi i m\tau_1}d\tau_1.$$

The explicit formulas for these coefficients are given by [Bum98, GM03]

$$a_0 = 2\zeta_R(2s)\tau_2^s + 2\phi(s)\zeta_R(2s-1)\tau_2^{1-s},$$

where

$$\phi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)},$$

and

$$a_n = 2\frac{\tau_2^{1/2}K_{s-1/2}(2\pi|n|\tau_2)}{\pi^{-s}\Gamma(s)}|n|^{s-1}\sigma_{1-2s}(n),$$

where for $n \geq 1$ and $v \in \mathbb{C}$, we let

$$\sigma_v(n) := \sum_{\substack{0<d\\ d|n}} d^v$$

denote the divisor function. In general, $\phi(s)$ is called the constant or scattering term and a_n with $n \geq 1$ is the non-trivial term.

In this chapter, we propose to recover these results by contour integration on the complex plane without Fourier techniques.

Indeed, the existence of two (or more) general methods of obtaining analytic continuations and functional equations of zeta functions has been known since Riemann's 1859 paper [Rie59] on the distribution of prime numbers (see [Tit86, §2]).

6.4 Eigenvalue problem set up

For fixed $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$ the general τ-Laplacian is

$$\Delta_\tau = -\frac{1}{\tau_2^2}\left[\left(\frac{\partial}{\partial x} + \tau_1\frac{\partial}{\partial y}\right)^2 + \left(\tau_2\frac{\partial}{\partial y}\right)^2\right] = -\frac{1}{\tau_2^2}\left[\frac{\partial^2}{\partial x^2} + (\tau_1^2+\tau_2^2)\frac{\partial^2}{\partial y^2} + 2\tau_1\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right]. \tag{6.29}$$

Now we let $M = S^1 \times S^1$ be a complex torus and the corresponding integral lattice is (see Fig. 6.3)

$$\mathcal{L}_\tau := \{a + b\tau \mid a, b \in \mathbb{Z}\}, \quad M := \mathbb{C}\backslash\mathcal{L}_\tau. \tag{6.30}$$

The relevant eigenvalue problem is

$$\Delta_\tau\phi_\lambda(x,y) = \lambda^2\phi_\lambda(x,y), \tag{6.31}$$

with periodic boundary conditions

$$\phi_\lambda(x,y) = \phi_\lambda(x+1,y), \quad \frac{\partial}{\partial x}\phi_\lambda(x,y) = \frac{\partial}{\partial x}\phi_\lambda(x+1,y), \tag{6.32}$$

on x, as well as

$$\phi_\lambda(x,y) = \phi_\lambda(x,y+1), \quad \frac{\partial}{\partial y}\phi_\lambda(x,y) = \frac{\partial}{\partial y}\phi_\lambda(x,y+1), \tag{6.33}$$

on y. By taking the eigenfunctions to be

$$\phi_{m,n}(x,y) = e^{-2\pi imx}e^{-2\pi iny}, \tag{6.34}$$

with $(m,n) \in \mathbb{Z}_*^2$, we see that

$$\begin{aligned}\Delta_\tau\phi_{m,n}(x,y) &= \frac{(2\pi)^2}{\tau_2^2}[m^2 + 2\tau_1 mn + (\tau_1^2+\tau_2^2)n^2]\phi_{m,n}(x,y)\\ &= \frac{(2\pi)^2}{\tau_2^2}\left|m+n\tau\right|^2\phi_{m,n}(x,y).\end{aligned}$$

Therefore, the eigenvalues to consider are

$$\lambda_{m,n}^2 = \frac{(2\pi)^2}{\tau_2^2}|m+n\tau|^2 = \frac{(2\pi)^2}{\tau_2^2}(m+n\tau)(m+n\bar{\tau}), \quad (m,n) \in \mathbb{Z}_*^2, \tag{6.35}$$

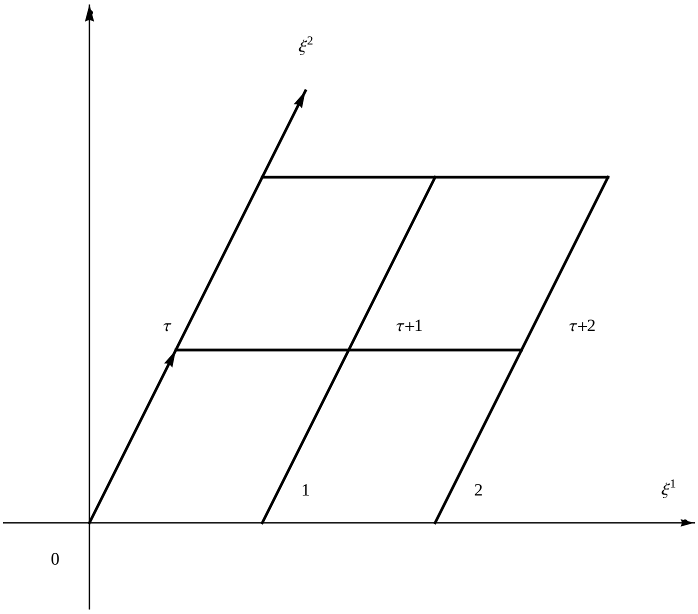

FIGURE 6.3: τ parametrizes the complex structure of this parallelogram [Nak03].

and we define the following spectral function.

Definition 6.6. For $\mathrm{Re}(s) > 1$ and $\tau \in \mathbb{H}$, the associated spectral zeta function of the general τ-Laplacian on the complex torus is defined to be

$$\zeta_{\Delta_\tau}(s) := \sum_{\lambda} (\lambda^2)^{-s} = (2\pi)^{-2s} \tau_2^{2s} \sum_{(m,n)\in\mathbb{Z}_*^2} |m + n\tau|^{-2s} = (2\pi)^{-2s} \tau_2^{s} E^*(s,\tau), \quad (6.36)$$

where $E^*(s,\tau)$ is the non-holomorphic Eisenstein series.

6.5 Main result

A majority of the cases treated in the literature of spectral zeta functions [Eli12, EOR$^+$94, Kir10] have eigenvalues which give rise to homogeneous Epstein zeta functions, of the type (6.26), with no mixed terms mn (or equivalently, where there is no mixed partial derivative $\partial^2/\partial x \partial y$ in the Laplacian). In particular, the inhomogeneous Epstein zeta function $\zeta_{\mathcal{E}}(s) = \sum_{(m,n)\in\mathbb{Z}_*^2} Q(m,n)^{-s}$ for a general quadratic form $Q(m,n) = am^2 + bmn + cn^2$ with $b \neq 0$ has not been computed with the argument principle, only with Poisson summation. This therefore constitutes a new application of the contour integration method which does not exist in the literature and where different insights are gained (in [Sie59] contour integration is used but the essential step is accomplished through Fourier methods). More general cases with $Q'(m,n) = am^2 + bmn + cn^2 + dm + en + f$, where f is a real positive constant, and in higher dimensions as well, have been treated in [Eli12] by means of Poisson summation.

We split the summation into $n = 0$, $m \in \mathbb{Z}\backslash\{0\}$ and $n \neq 0$, $m \in \mathbb{Z}$. Thus we write

$$\zeta_{\Delta_\tau}(s) = (2\pi)^{-2s} \tau_2^{2s} \sideset{}{'}\sum_{m=-\infty}^{\infty} m^{-2s} + (2\pi)^{-2s} \tau_2^{2s} \sideset{}{'}\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [(m + n\tau)(m + n\bar{\tau})]^{-s}$$

$$:= 2(2\pi)^{-2s}\tau_2^{2s}\zeta_R(2s) + (2\pi)^{-2s}\tau_2^{2s}\zeta_I(s),$$

where $\zeta_R(s)$ denotes the Riemann zeta function and

$$\zeta_I(s) = \sum_{n=-\infty}^{\infty}{}' \sum_{m=-\infty}^{\infty} [(m+n\tau)(m+n\bar{\tau})]^{-s}.$$

Summation with a prime indicates that we remove the term corresponding to $n = 0$ and $m = 0$. We represent the zeta function $\zeta_I(s)$ in terms of a contour integral; the summation over m is expressed using $\sin(\pi k) = 0$. Thus, we write

$$\zeta_I(s) = \sum_{n=-\infty}^{\infty}{}' \int_\gamma \frac{dk}{2\pi i}[(k+n\tau)(k+n\bar{\tau})]^{-s}\frac{d}{dk}\log\sin(\pi k),$$

where γ is the contour in Fig. 6.4. When deforming the contour, we need to know the

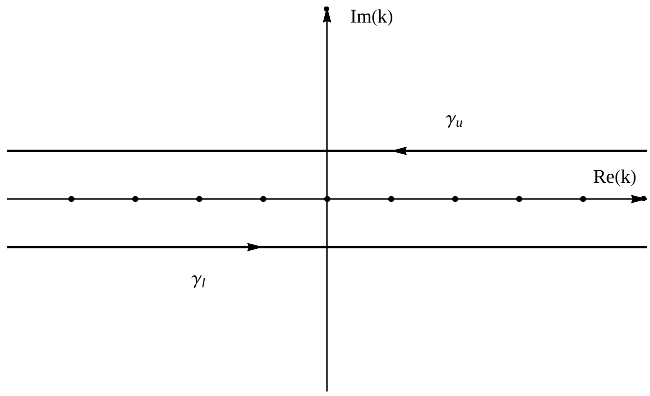

FIGURE 6.4: Contour γ enclosing the eigenvalues.

location of the branch cuts. So we solve for

$$(k+n\tau)(k+n\bar{\tau}) = -\alpha,$$

where $\alpha \in \mathbb{R}$ and $\alpha \geq 0$. Writing $\tau = \tau_1 + i\tau_2$, we have

$$k^2 + n^2|\tau|^2 + kn\bar{\tau} + kn\tau + \alpha = 0, \tag{6.37}$$

so that

$$k = -n\tau_1 \pm \sqrt{n^2(\tau_1^2 - |\tau|^2) - \alpha} = -n\tau_1 \pm \sqrt{-n^2\tau_2^2 - \alpha}.$$

We notice that

$$-n^2\tau_2^2 - \alpha \leqslant 0,$$

and the branch cuts are something like the ones given in Fig. 6.5. Let us denote the branch points by z_b^n and $\bar{z}_b^n$, thus

$$z_b^n = -n\tau_1 + i\sqrt{n^2\tau_2^2} = -n\tau_1 + i\,|n|\,\tau_2.$$

The natural deformation is, therefore, as indicated in Fig. 6.6. When shrinking the contours to the branch cuts, the parametrizations will be done according to the following:

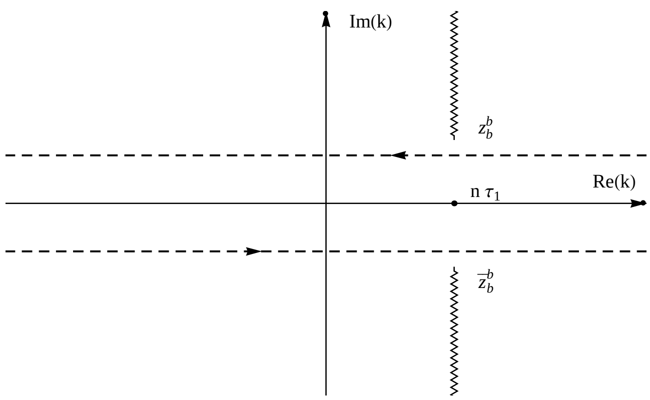

FIGURE 6.5: Location of the branch points

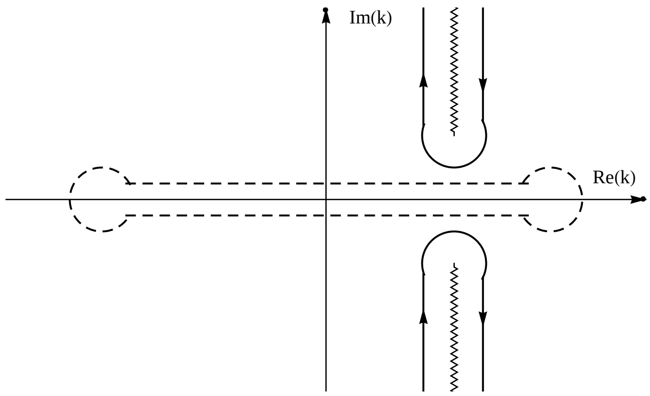

FIGURE 6.6: Newly deformed contour enclosing the branch cuts

1. for the upper contour

$$k = z_b^n + e^{i\pi/2}u, \quad u \in (\infty, 0],$$
$$k = z_b^n + e^{-3i\pi/2}u, \quad u \in [0, \infty);$$

2. similarly for the lower contour

$$k = \bar{z}_b^n + e^{-i\pi/2}u, \quad u \in [0, \infty),$$
$$k = \bar{z}_b^n + e^{3i\pi/2}u, \quad u \in (\infty, 0].$$

For $\zeta_I(s)$ this gives

$$\begin{aligned}\zeta_I(s) = \sum_{n=-\infty}^{\infty}{}' \Bigg\{ &\int_\infty^0 \frac{du}{2\pi i}[(z_b^n + e^{i\pi/2}u + n\tau)(z_b^n + e^{i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du}\log\sin(\pi[z_b^n + e^{i\pi/2}u]) \\ &+ \int_0^\infty \frac{du}{2\pi i}[(z_b^n + e^{-3i\pi/2}u + n\tau)(z_b^n + e^{-3i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du}\log\sin(\pi[z_b^n + e^{-3i\pi/2}u]) \\ &+ \int_0^\infty \frac{du}{2\pi i}[(\bar{z}_b^n + e^{-i\pi/2}u + n\tau)(\bar{z}_b^n + e^{-i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du}\log\sin(\pi[\bar{z}_b^n + e^{-i\pi/2}u]) \\ &+ \int_\infty^0 \frac{du}{2\pi i}[(\bar{z}_b^n + e^{3i\pi/2}u + n\tau)(\bar{z}_b^n + e^{3i\pi/2}u + n\bar{\tau})]^{-s} \frac{d}{du}\log\sin(\pi[\bar{z}_b^n + e^{3i\pi/2}u]) \Bigg\}.\end{aligned}$$

We next rewrite the integrands using the fact that z_b^n and $\bar{z}_b^n$ solve the quadratic equation (6.37) with $\alpha = 0$. We then use the notation

$$\zeta_I(s) = \zeta_I^{(1)}(s) + \zeta_I^{(2)}(s) + \zeta_I^{(3)}(s) + \zeta_I^{(4)}(s)$$

to denote each individual (infinite) sum above. Let us start with $\zeta_I^{(1)}(s)$: we begin by computing

$$(z_b^n + e^{i\pi/2}u + n\tau)(z_b^n + e^{i\pi/2}u + n\bar{\tau}) = e^{i\pi}(u^2 + 2u\,|n|\,\tau_2),$$

so that we have

$$\zeta_I^{(1)}(s) = \sum_{n=-\infty}^{\infty}{}' \,(-e^{-i\pi s}) \int_0^\infty \frac{du}{2\pi i}(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log\sin(\pi[z_b^n + iu]).$$

Redoing the same computation for $\zeta_I^{(2)}(s)$, but replacing $e^{i\pi/2}$ with $e^{-3i\pi/2}$ yields

$$(z_b^n + e^{-3i\pi/2}u + n\tau)(z_b^n + e^{-3i\pi/2}u + n\bar{\tau}) = e^{-i\pi}(u^2 + 2u\,|n|\,\tau_2),$$

so that

$$\zeta_I^{(2)}(s) = \sum_{n=-\infty}^{\infty}{}' \, e^{i\pi s} \int_0^\infty \frac{du}{2\pi i}(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log\sin(\pi[z_b^n + iu])$$

and, combining the two terms,

$$\zeta_I^{(1)}(s) + \zeta_I^{(2)}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty du(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log\sin(\pi[z_b^n + iu]).$$

We next consider the log terms in order to perform the analytic continuation. A suitable rewriting is

$$\sin(\pi[z_b^n + iu]) = \frac{e^{i\pi(iu+z_b^n)} - e^{-i\pi(iu+z_b^n)}}{2i} = -\frac{1}{2i} e^{\pi u - i\pi z_b^n}(1 - e^{-2\pi u + 2i\pi z_b^n}).$$

Note that $e^{2i\pi z_b^n} = e^{2i\pi(-n\tau_1 + i|n|\tau_2)}$ is exponentially damped for large $|n|$. We therefore write

$$\begin{aligned}
\zeta_I^{(1)}(s) + \zeta_I^{(2)}(s) &= \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty \frac{du}{(u^2 + 2u\,|n|\,\tau_2)^s} \frac{d}{du} \log[e^{\pi u - i\pi z_b^n}(1 - e^{-2\pi u + 2i\pi z_b^n})] \\
&= \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty \frac{du}{(u^2 + 2u\,|n|\,\tau_2)^s} \left\{\pi + \frac{d}{du}\log(1 - e^{-2\pi u + 2i\pi z_b^n})\right\} \\
&=: \zeta_I^{(12,1)}(s) + \zeta_I^{(12,2)}(s),
\end{aligned}$$

where we define

$$\zeta_I^{(12,1)}(s) := \sin(\pi s) \sum_{n=-\infty}^{\infty}{}' \int_0^\infty du(u^2 + 2u\,|n|\,\tau_2)^{-s},$$

and

$$\zeta_I^{(12,2)}(s) := \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty du(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log(1 - e^{-2\pi u + 2i\pi z_b^n}).$$

To compute the derivative at $s = 0$ we only need to find the analytical continuation of $\zeta_I^{(12,1)}(s)$ since $\zeta_I^{(12,2)}(s)$ is already valid for all $s \in \mathbb{C}$. In order to accomplish this continuation we note that, for $\frac{1}{2} < \mathrm{Re}(s) < 1$,

$$\int_0^\infty du\ u^{-s}(u + 2x)^{-s} = x^{1-2s} \frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}}. \tag{6.38}$$

With this in mind, we write

$$\begin{aligned}\zeta_I^{(12,1)}(s) &= \sin(\pi s) \sum_{n=-\infty}^{\infty}{}' \int_0^\infty \frac{du}{(u^2 + 2u\,|n|\,\tau_2)^s} = 2\sin(\pi s) \sum_{n=1}^{\infty} \int_0^\infty du u^{-s}(u + 2n\tau_2)^{-s} \\ &= 2\sin(\pi s) \frac{\Gamma(1-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}} \tau_2^{1-2s} \sum_{n=1}^{\infty} n^{1-2s} = \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} \tau_2^{1-2s} \zeta_R(2s-1).\end{aligned}$$

The branch in the lower half-plane is handled accordingly. In order to simplify the integrand, we note the analogy between the first and third case: $e^{i\pi/2} \to e^{-i\pi/2}$, $z_b^n \to \bar{z}_b^n$. Thus

$$(\bar{z}_b^n + e^{-i\pi/2}u + n\tau)(\bar{z}_b^n + e^{-i\pi/2}u + n\bar{\tau}) = e^{-i\pi}(u^2 + 2u\,|n|\,\tau_2),$$

so that the third function is

$$\zeta_I^{(3)}(s) = \sum_{n=-\infty}^{\infty}{}' (e^{i\pi s}) \int_0^\infty \frac{du}{2\pi i}(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log\sin(\pi[\bar{z}_b^n - iu]).$$

Similarly, for the last function it follows that

$$\zeta_I^{(4)}(s) = \sum_{n=-\infty}^{\infty}{}' (-e^{-i\pi s}) \int_0^\infty \frac{du}{2\pi i}(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log\sin(\pi[\bar{z}_b^n - iu]).$$

Adding up these two terms yields

$$\zeta_I^{(3)}(s) + \zeta_I^{(4)}(s) = \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty du(u^2 + 2u\,|n|\,\tau_2)^{-s} \frac{d}{du} \log\sin(\pi[\bar{z}_b^n - iu]).$$

Going through a similar manipulation of the log sin term as above allows us to write

$$\sin(\pi[\bar{z}_b^n - iu]) = \frac{e^{i\pi(\bar{z}_b^n - iu)} - e^{-i\pi(\bar{z}_b^n - iu)}}{2i} = \frac{1}{2i} e^{\pi u + i\pi\bar{z}_b^n}(1 - e^{-2\pi u - 2i\pi\bar{z}_b^n}).$$

We note that $\pi u + i\pi\bar{z}_b^n = \pi u + i\pi(-n\tau_1 - i\,|n|\,\tau_2)$ and so $e^{-2\pi u - 2i\pi\bar{z}_b^n}$ is, like in the previous case, exponentially damped as $|n| \to \infty$. Once more, using (6.38),

$$\begin{aligned}\zeta_I^{(3)}(s) + \zeta_I^{(4)}(s) &= \frac{\sin(\pi s)}{\pi} \sum_{n=-\infty}^{\infty}{}' \int_0^\infty du(u^2 + 2u\,|n|\,\tau_2)^{-s} \left[\pi + \frac{d}{du} \log(1 - e^{-2\pi u - 2i\pi\bar{z}_b^n})\right] \\ &= \frac{\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)} \tau_2^{1-2s} \zeta_R(2s-1)\end{aligned}$$

$$+\frac{\sin(\pi s)}{\pi}\sum_{n=-\infty}^{\infty}{}' \int_0^\infty du(u^2+2u\,|n|\,\tau_2)^{-s}\frac{d}{du}\log(1-e^{-2\pi u-2i\pi\bar{z}_b^n}).$$

Therefore, the final result is

$$\begin{aligned}\zeta_{\Delta_\tau}(s) &= 2(2\pi)^{-2s}\tau_2^{2s}\zeta_R(2s)+(2\pi)^{1-2s}\tau_2^{2s}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}\tau_2^{1-2s}\zeta_R(2s-1)\\ &+(2\pi)^{-2s}\tau_2^{2s}\frac{\sin(\pi s)}{\pi}\sum_{n=-\infty}^{\infty}{}'\int_0^\infty duu^{-s}(u+2\,|n|\,\tau_2)^{-s}\\ &\times\frac{d}{du}\log[(1-e^{-2\pi u+2i\pi z_b^n})(1-e^{-2\pi u+2i\pi\bar{z}_b^n})],\end{aligned}$$

which is now valid for all $s\in\mathbb{C}\backslash\{1\}$. Re-writing the sum so that it goes from $n=1$ to $n=\infty$ we have thus proved the following result.

Theorem 6.7. *The spectral zeta function of Δ_τ on $S^1\times S^1$ can be written as*

$$\begin{aligned}\zeta_{\Delta_\tau}(s) &= 2(2\pi)^{-2s}\tau_2^{2s}\zeta_R(2s)+(2\pi)^{1-2s}\tau_2\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}\zeta_R(2s-1)\\ &+\frac{2\sin(\pi s)}{\pi}\left(\frac{2\pi}{\tau_2}\right)^{-2s}\sum_{n=1}^{\infty}\int_0^\infty du(u^2+2un\tau_2)^{-s}\\ &\times\frac{d}{du}\log[(1-e^{-2\pi u-2i\pi n\bar{\tau}})(1-e^{-2\pi u+2i\pi n\tau})],\end{aligned}\tag{6.39}$$

for $\tau\in\mathbb{H}$ and $s\in\mathbb{C}\backslash\{1\}$.

Before proceeding to explain the term on the second line of (6.39), we first compute the functional determinant we were interested in. The derivative of the last expression at $s=0$ is obtained in terms of the Dedekind eta function:

$$\begin{aligned}\zeta'_{\Delta_\tau}(0) &= -\log\tau_2^2+\frac{\pi\tau_2}{3}+2\sum_{n=1}^{\infty}\int_0^\infty du\frac{d}{du}\log[(1-e^{-2\pi u-2i\pi n\bar{\tau}})(1-e^{-2\pi u+2i\pi n\tau})]\\ &= -\log\tau_2^2+\frac{\pi\tau_2}{3}-2\sum_{n=1}^{\infty}\left[\log(1-e^{-2i\pi n\bar{\tau}})+\log(1-e^{-2\pi u+2i\pi n\tau})\right]\\ &= -\log\tau_2^2+\frac{\pi\tau_2}{3}-2\left(\log\eta(\tau)-\frac{\pi i\tau}{12}+\log\eta(-\bar{\tau})+\frac{\pi i\bar{\tau}}{12}\right)\\ &= -\log\tau_2^2+\frac{\pi\tau_2}{3}-2\left(\log[\eta(\tau)\eta(-\bar{\tau})]+\frac{\pi i}{12}(\bar{\tau}-\tau)\right)\\ &= -\log(\tau_2^2|\eta(\tau)|^4),\end{aligned}$$

since $\eta(-\bar{\tau})=\overline{\eta(\tau)}$. Therefore, we have the following result [DKE+00, FMS97, Pol98, Wil10].

Theorem 6.8. *The functional determinant of the τ-Laplacian on the complex torus is*

$$\det(\Delta_\tau)=\tau_2^2|\eta(\tau)|^4.\tag{6.40}$$

Proof. By definition

$$\det(\Delta_\tau)=\exp(-\zeta'_{\Delta_\tau}(0))=\tau_2^2|\eta(\tau)|^4,$$

as claimed. □

We note that in order to obtain the value of $\zeta'_{\Delta_\tau}(0)$ we have not used Kronecker's first limit formula, which in turn, depends on the functional equation of $E^*(s,\tau)$. Thus, viewed under this optic, the method of contour integration is cheaper in the sense that it requires less resources to provide the functional determinant. For a derivation of this theorem using the Kronecker formula, see, for instance, [Wil10].

6.6 Additional results and consequences

6.6.1 The Chowla-Selberg series formula

In fact, the term on the second line of (6.39) can be shown to be the term from the Chowla-Selberg series formula obtained through Poisson summation methods. To see this, let us compute the log-terms further. Expanding the logarithm,

$$\log(1-e^{-2\pi u-2i\pi n\bar{\tau}}) = -\sum_{k=1}^{\infty}\frac{1}{k}e^{-2\pi ku-2i\pi nk\bar{\tau}},$$

and for $\mathrm{Re}(s)<1$ the integral becomes

$$\begin{aligned}
\Upsilon_{\bar{\tau}}(s,n) &:= \int_0^\infty du u^{-s}(u+2n\tau_2)^{-s}\frac{d}{du}\log[(1-e^{-2\pi u-2i\pi n\bar{\tau}})] \\
&= -\int_0^\infty du u^{-s}(u+2n\tau_2)^{-s}\frac{d}{du}\sum_{k=1}^{\infty}\frac{1}{k}e^{-2\pi ku-2i\pi nk\bar{\tau}} \\
&= 2\pi\sum_{k=1}^{\infty}\int_0^\infty du u^{-s}(u+2n\tau_2)^{-s}e^{-2\pi ku-2i\pi nk\bar{\tau}} \\
&= 2\pi\sum_{k=1}^{\infty}e^{-2i\pi nk\bar{\tau}}\int_0^\infty du u^{-s}(u+2n\tau_2)^{-s}e^{-2\pi ku} \\
&= 2\pi\sum_{k=1}^{\infty}e^{-2i\pi nk\bar{\tau}}\frac{K_{1/2-s}(2\pi kn\tau_2)}{\sin(\pi s)\Gamma(s)}e^{2\pi nk\tau_2}k^{s-1/2}\pi^s\frac{1}{\sqrt{n\tau_2}}(n\tau_2)^{1-s} \\
&= \frac{2\pi^{1+s}\tau_2^{1/2-s}}{\sin(\pi s)\Gamma(s)}\sum_{k=1}^{\infty}e^{-2\pi nki\tau_1}k^{s-1/2}n^{1/2-s}K_{1/2-s}(2\pi kn\tau_2),
\end{aligned}$$

where $K_n(z)$ is the modified Bessel function of the second kind and where we have used the Mellin inversion of equation (10.32.13) of [OM14]. Moreover, the integral of the other log-term only differs in that $e^{-2\pi ink\bar{\tau}}$ is replaced by $e^{2\pi ink\tau}$. Thus, noting that $\tau_2+i\tau=\tau_2+i(\tau_1+i\tau_2)=i\tau_1$, we have

$$\begin{aligned}
\Upsilon_{\tau}(s,n) &:= \int_0^\infty du u^{-s}(u+2n\tau_2)^{-s}\frac{d}{du}\log[(1-e^{-2\pi u+2i\pi n\tau})] \\
&= \frac{2\pi^{1+s}\tau_2^{1/2-s}}{\sin(\pi s)\Gamma(s)}\sum_{k=1}^{\infty}e^{2\pi nki\tau_1}k^{s-1/2}n^{1/2-s}K_{1/2-s}(2\pi kn\tau_2).
\end{aligned}$$

If we set the third term in (6.39) to be $Q(s,\tau)$, then we see that

$$\begin{aligned}
Q(s,\tau) &= \frac{2\sin(\pi s)}{\pi}\left(\frac{2\pi}{\tau_2}\right)^{-2s}\sum_{n=1}^{\infty}\left[\Upsilon_\tau(s,n)+\Upsilon_{\bar\tau}(s,n)\right]\\
&= \frac{2\sin(\pi s)}{\pi}\left(\frac{2\pi}{\tau_2}\right)^{-2s}\frac{2\pi^{1+s}\tau_2^{1/2-s}}{\sin(\pi s)\Gamma(s)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{2\pi nki\tau_1}k^{s-1/2}n^{1/2-s}K_{1/2-s}(2\pi kn\tau_2)\\
&+ \frac{2\sin(\pi s)}{\pi}\left(\frac{2\pi}{\tau_2}\right)^{-2s}\frac{2\pi^{1+s}\tau_2^{1/2-s}}{\sin(\pi s)\Gamma(s)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{-2\pi nki\tau_1}k^{s-1/2}n^{1/2-s}K_{1/2-s}(2\pi kn\tau_2)\\
&= \frac{2^{3-2s}\pi^{-s}\tau_2^{1/2+s}}{\Gamma(s)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\cos(2\pi nk\tau_1)\left(\frac{k}{n}\right)^{s-1/2}K_{1/2-s}(2\pi kn\tau_2).
\end{aligned}$$

The key is now to relate the expression inside the double sum to the divisor function. The main property we are interested in is the following way of changing the double sum for a single sum while bringing in the divisor function

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}e^{\pm 2\pi ikn\tau_1}K_{1/2-s}(2\pi kn\tau_2)\left(\frac{k}{n}\right)^{s-1/2}=\sum_{n=1}^{\infty}\sigma_{1-2s}(n)e^{\pm 2\pi in\tau_1}K_{1/2-s}(2\pi n\tau_2)n^{s-1/2}.$$

For a proof of a more general result of this type, see for instance [Wil10]. Thus, switching back to $E^*(s,\tau)$ instead of $\zeta_{\Delta_\tau}(s)$, we have arrived at the following theorem, which now holds for all $s\neq 1$ by analytic continuation [CS67].

Theorem 6.9 (Chowla-Selberg series formula)**.** *One has*

$$\begin{aligned}
E^*(s,\tau) &= 2\tau_2^s\zeta_R(2s)+2\pi\tau_2^{1-s}\frac{\Gamma(s-\frac{1}{2})}{\sqrt{\pi}\Gamma(s)}\zeta_R(2s-1)\\
&+\frac{8\pi^s\tau_2^{1/2}}{\Gamma(s)}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\cos(2\pi n\tau_1)K_{1/2-s}(2\pi n\tau_2)n^{s-1/2},
\end{aligned}$$

for $\tau\in\mathbb{H}$ *and* $s\in\mathbb{C}\backslash\{1\}$*.*

6.6.2 The Nan-Yue Williams formula

As a byproduct, we may now equate the two 'remainders' in the above expressions of $E^*(s,\tau)$, recalling that the one in (6.39) has to be multiplied by $(2\pi)^{2s}\tau_2^{-s}$, to get

$$\begin{aligned}
Q(s,\tau) &= \frac{8\pi^s\tau_2^{1/2}}{\Gamma(s)}\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\cos(2\pi n\tau_1)K_{1/2-s}(2\pi n\tau_2)n^{s-1/2}\\
&= 2\tau_2^s\frac{\sin(\pi s)}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}du(u^2+2un\tau_2)^{-s}\frac{d}{du}\log[(1-e^{-2\pi u-2i\pi n\bar\tau})(1-e^{-2\pi u+2i\pi n\tau})].
\end{aligned}$$

Using the Euler reflection formula for the Γ-function we obtain

$$\sum_{n=1}^{\infty}\sigma_{1-2s}(n)\cos(2\pi n\tau_1)K_{1/2-s}(2\pi n\tau_2)n^{s-1/2}=\frac{\tau_2^{-1/2}}{4}\left(\frac{\tau_2}{\pi}\right)^s\frac{1}{\Gamma(1-s)}$$

$$\times \sum_{n=1}^{\infty} \int_0^{\infty} du(u^2 + 2un\tau_2)^{-s} \frac{d}{du} \log[(1 - e^{-2\pi u - 2i\pi n\bar{\tau}})(1 - e^{-2\pi u + 2i\pi n\tau})]. \tag{6.41}$$

At $s = 0$ the integral is easily evaluated by a similar computation to the one done previously for $\zeta'_{\Delta_\tau}(0)$, so that we are left with the following consequence.

Corollary 6.10. *For $\tau \in \mathbb{H}$, one has*

$$\sum_{n=1}^{\infty} \sigma_1(n) \cos(2\pi n\tau_1) K_{1/2}(2\pi n\tau_2) n^{-1/2} = -\frac{\tau_2^{-1/2}}{2} \log|\eta(\tau)| - \frac{\tau_2^{1/2}\pi}{24}.$$

In [NYW95] this is given in a slightly different context and

$$\sum_{n=1}^{\infty} \sigma_1(n) K_{1/2}(2\pi n) n^{-1/2} = -\frac{1}{2} \log \eta(i) - \frac{\pi}{24} \approx 0.000936341.$$

for the particular case $\tau = i$.

6.6.3 The Kronecker first limit formula

Let us now go back to the Chowla-Selberg formula. Using the functional equation of the divisor function and the Bessel function, respectively,

$$\sigma_\nu(n) = n^\nu \sigma_{-\nu}(n), \quad K_\nu(x) = K_{-\nu}(x),$$

it is not difficult to show that the following identity for $E^*(s,\tau)$ holds

$$\pi^{1-2s}\Gamma(s)E^*(s,\tau) = \Gamma(1-s)E^*(1-s,\tau). \tag{6.42}$$

Using this functional equation and the result of the determinant of the Laplacian (6.40) we can now reverse the steps of the proof of Kronecker's first limit formula [Wil10]. All known proofs (e.g. [Mot68, NYW95, Sie59, Shi80a, Wil10]) use a variation of some kind of Poisson summation. The present one is a new proof in the sense that no techniques from Fourier analysis are ever used.
Interesting steps in the direction of special functions and complex integration were worked out in [Shi80a, Var88] using the theory of multiple Gamma functions and Barnes zeta functions. Furthermore, using the Barnes double gamma function and the Selberg zeta function, the determinants of the n-sphere and spinor fields on a Riemann surface are found in [Sar87]; see also [Dow94a, Dow94b].

Proof of Theorem 6.5. First we define

$$f(s) := \Gamma(s)\pi^{2-2s}/\Gamma(2-s), \tag{6.43}$$

so that

$$f(1) = 1, \quad \lim_{s\to 1} \frac{f(s) - f(1)}{s-1} = f'(1) = -2\log\pi - 2\gamma, \tag{6.44}$$

where γ is Euler's constant. The key idea is to use the result we already got

$$\zeta'_{\Delta_\tau}(0) = -\log \tau_2^2 \, |\eta(\tau)|^4 \,, \tag{6.45}$$

from which

$$\begin{aligned}
\left.\frac{\partial}{\partial s}\right|_{s=0} E^*(s,\tau) &= \left.\frac{(2\pi)^{2s}}{\tau_2^s}\right|_{s=0} \zeta'_{\Delta_\tau}(0) + \left.\frac{\partial}{\partial s}\right|_{s=0} \left(\frac{(2\pi)^{2s}}{\tau_2^s}\right) \zeta_{\Delta_\tau}(0) \\
&= -\log(\tau_2^2 \, |\eta(\tau)|^4) - (\log 4\pi^2 - \log \tau_2) \\
&= -\log(4\pi^2 \tau_2 \, |\eta(\tau)|^4),
\end{aligned} \tag{6.46}$$

and confront it with the definition of the derivative of $E^*(s,\tau)$,

$$\begin{aligned}
\left.\frac{\partial}{\partial s}\right|_{s=0} E^*(s,\tau) := \lim_{s\to 0} \frac{E^*(s,\tau) - E^*(0,\tau)}{s-0} &= \lim_{s\to 1} \frac{E^*(1-s,\tau)+1}{1-s} \\
&= \lim_{s\to 1} \left(\frac{E^*(1-s,\tau)\Gamma(s)}{\Gamma(s)(1-s)} - \frac{1}{s-1} \right) \\
&= \lim_{s\to 1} \left(\frac{\Gamma(s)}{1-s} \pi^{1-2s} \frac{E^*(s,\tau)}{\Gamma(1-s)} - \frac{1}{s-1} \right) \\
&= \lim_{s\to 1} \left[\frac{\Gamma(s)\pi^{1-2s}}{\Gamma(2-s)} \left(E^*(s,\tau) - \frac{\pi}{s-1} \right) + \frac{\Gamma(s)\pi^{1-2s}}{\Gamma(2-s)} \frac{\pi}{s-1} - \frac{1}{s-1} \right] \\
&= \lim_{s\to 1} \frac{\Gamma(s)\pi^{1-2s}}{\Gamma(2-s)} \left(E^*(s,\tau) - \frac{\pi}{s-1} \right) + \lim_{s\to 1} \frac{f(s)-f(1)}{s-1} \\
&= \frac{1}{\pi} \lim_{s\to 1} \left(E^*(s,\tau) - \frac{\pi}{s-1} \right) - 2\log\pi - 2\gamma.
\end{aligned} \tag{6.47}$$

Comparing (6.46) and (6.47) yields

$$\frac{1}{\pi} \lim_{s\to 1} \left(E^*(s,\tau) - \frac{\pi}{s-1} \right) - 2\log\pi - 2\gamma = -\log(4\pi^2 \tau_2 \, |\eta(\tau)|^4)$$

and with some rearrangements

$$\lim_{s\to 1} \left(E^*(s,\tau) - \frac{\pi}{s-1} \right) = 2\pi(\gamma - \log 2 - \log \tau_2^{1/2} |\eta(\tau)|^2),$$

which is the desired result. □

6.6.4 The Lambert series

We may re-write (6.41) as

$$\begin{aligned}
\mathcal{Q}(s,\tau) &:= \sum_{n=1}^{\infty} \sigma_{1-2s}(n) \cos(2\pi n \tau_1) K_{1/2-s}(2\pi n \tau_2) n^{s-1/2} \\
&= \frac{\tau_2^{-1/2}}{4} \left(\frac{\tau_2}{\pi}\right)^s \frac{1}{\Gamma(2-s)} \sum_{n=1}^{\infty} \int_0^\infty \frac{dx}{2x + 2n\tau_2} \left[\frac{d}{dx} (x^2 + 2xn\tau_2)^{-s+1} \right] \\
&\qquad \times \frac{d}{dx} \log[(1 - e^{-2\pi x - 2\pi i n \bar{\tau}})(1 - e^{-2\pi x + 2\pi i n \tau})],
\end{aligned}$$

so that, after integrating by parts, we have

$$\mathcal{Q}(s,\tau) = \frac{\tau_2^{-1/2}}{4}\left(\frac{\tau_2}{\pi}\right)^s \frac{1}{\Gamma(2-s)} \sum_{n=1}^{\infty} \Bigg[\frac{2\pi(1-e^{2\pi i n\tau}+e^{2\pi i n(\tau+\bar\tau)})}{(e^{2\pi i n\tau}-1)(e^{2\pi i n\bar\tau}-1)} \times \frac{\Gamma(2-s)\Gamma(s-\frac{1}{2})}{2\sqrt{\pi}}(n\tau_2)^{1-2s} + \int_0^\infty dx\, g(s,x,n,\tau_2)\frac{d^2}{dx^2}\log[(1-e^{-2\pi x-2\pi i n\bar\tau})(1-e^{-2\pi x+2\pi i n\tau})\Bigg]$$

where

$$g(s,x,n,\tau_2) := x\ {}_2F_1\left(1-s,s,2-s,-\frac{x}{2n\tau_2}\right)\left(1+\frac{x}{2n\tau_2}\right)^s (x(x+2n\tau_2))^{-s},$$

with ${}_2F_1$ the usual hypergeometric function. Setting $s=1$, we have

$$g(1,x,n,\tau_2) = \frac{1}{2n\tau_2}$$

and the integral can, once again, be easily evaluated

$$\mathcal{Q}(1,\tau) = -\frac{\tau_2^{-1/2}}{4}\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1-2e^{2\pi i n\tau}+e^{2\pi i n(\tau+\bar\tau)})}{(e^{2\pi i n\tau}-1)(e^{2\pi i n\bar\tau}-1)}.$$

The case $\tau = i$ yields

$$\mathcal{Q}(1,i) = \sum_{n=1}^{\infty}\sigma_{-1}(n)K_{-1/2}(2\pi n)n^{1/2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{e^{2\pi n}-1} \approx 0.000936341.$$

Using the fact that

$$K_{\pm 1/2}(2\pi n) = \frac{1}{2\sqrt{n}}e^{-2\pi n},$$

we obtain

$$\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2\pi n} = \sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{e^{2\pi n}-1},$$

which is a special case of the Lambert series [Sie59]

$$\sum_{n=1}^{\infty}\sigma_\alpha(n)q^n = \sum_{n=1}^{\infty}\frac{n^\alpha q^n}{1-q^n},$$

with $\alpha = -1$ and $q = e^{-2\pi}$. To obtain a different $\sigma_\alpha(n)$ term, with $\alpha \in -\mathbb{Z}_*$ we have to perform additional processes of integration by parts.

Appendix A

Appendix

A.1 Special functions

The goal of this brief appendix is to introduce the notation and early properties of some special functions used widely in this thesis. The most common special functions are extensively described in [AS72, AH12, Tit68, WW62] and the Riemann zeta-function can be picked up from [Edw74, Ivi85, Tit86]. Specific Mellin transforms can be found in [Obe74].

Although something has been lost by making the exposition of these functions so brief, a counterpart has been gained by devoting additional space to original research in the previous chapters.

A.1.1 The gamma function Γ

The gamma function has been studied extensively. It is the extension of the factorial function so that $\Gamma(n) = (n-1)!$. Its most important properties as listed below. It has the integral representation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{A.1}$$

for $\mathrm{Re}(z) > 0$. It is analytic everywhere except for simple poles at $z = 0, -1, -2, \cdots$ where it has residue

$$\operatorname*{res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!} \tag{A.2}$$

for $k = 0, 1, 2, \cdots$. It also satisfies the functional equation

$$\Gamma(1+z) = z\Gamma(z), \tag{A.3}$$

and reflection formula

$$\Gamma(z)\Gamma(1+z) = \frac{\pi}{\sin(\pi x)}, \tag{A.4}$$

due to Euler. Some of its special values are

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad \text{and} \quad \Gamma'(1) = -\gamma, \tag{A.5}$$

where γ is the Euler constant. Stirling's formula for $\Gamma(s)$ with $s = \sigma + it$, in a vertical strip $c \leq \sigma \leq d$ is given by [Cop35, p. 224]

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}} e^{-\frac{1}{2}\pi|t|}\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{A.6}$$

as $|t| \to \infty$. Stirling's formula in exact form [dB25, p. 47] reads

$$\Gamma(s) = \sqrt{2\pi} e^{-s} s^{s-\frac{1}{2}} \exp(O(|s|^{-1})), \tag{A.7}$$

as $|t| \to \infty$.

A.1.2 The Bessel functions J, Y and K

The Bessel functions of the first kind $J_\nu(x)$ or order ν are defined as solutions to the Bessel differential equation

$$x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{A.8}$$

which are non-singular the origin. They accept the representation

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu + n + 1)}. \tag{A.9}$$

The Bessel function of the second kind $Y_\nu(x)$ is defined as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \tag{A.10}$$

for non-integer ν. In the case when ν is an integer, then the function is defined by taking the limit

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x). \tag{A.11}$$

The modified Bessel function of the first kind which is defined by

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+\nu}}{m!\Gamma(m + \nu + 1)}. \tag{A.12}$$

Finally, the modified Bessel function of the second kind $K_\nu(x)$ is defined for non-integer ν as

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}, \tag{A.13}$$

when ν is not an integer. When ν is an integer, then a similar limit as the one above is used.

A.1.3 The hypergeometric functions

The Gaussian hypergeometric function ${}_2F_1$ is defined for $|z| < 1$ as

$$ {}_2F_1(q, b, c; z) = \sum_{n=0}^{\infty} \frac{(q)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{A.14} $$

It is undefined or infinite when c is equal to a non-positive integer. The symbol $(q)_n$ is the Pochhammer symbol defined by

$$ (q)_n = \begin{cases} 1, & \text{if } n = 0, \\ q(q+1)\cdots(q+n-1), & \text{if } n > 0. \end{cases} $$

One of its most important properties was derived by Gauss and it states that

$$ {}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{A.15} $$

with $\mathrm{Re}(c) > \mathrm{Re}(a+b)$. The confluent hypergeometric function ${}_1F_1$ has hypergeometric series given by

$$ {}_1F_1(a, b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}. \tag{A.16} $$

One of its essential features is the integral representation

$$ {}_1F_1(a, b; z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \tag{A.17} $$

In general, the general hypergeometric function is given by a hypergeometric series, i.e. a series for which the ratio of successive terms can be written as

$$ \frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}. \tag{A.18} $$

The resulting generalized hypergeometric function is written

$$ \sum_{k=0}^{\infty} c_k x^k = {}_pF_q \left[\begin{matrix} a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} ; x \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}. \tag{A.19} $$

The only other special case we will be needing is ${}_1F_2$.

A.1.4 The Lommel functions

The Lommel functions s and S are defined by

$$ s_{\mu,\nu}(z) = z^{\mu+1} \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{z}{2})^{2m+2} \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2}) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2})}{\Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu + m + \frac{3}{2}) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + m + \frac{1}{2})} \tag{A.20} $$

where $\mu \pm \nu$ is not a negative odd integer, and

$$ S_{\mu,\nu}(z) = s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2}) \Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2}) $$

$$\times \frac{\cos[\frac{1}{2}(\mu-\nu)\pi]J_{-\nu}(z) - \cos[\frac{1}{2}(\mu+\nu)\pi]J_{\nu}(z)}{\sin(\nu\pi)}.$$

A.1.5 The logarithmic and exponential integrals

The exponential integral is

$$\mathrm{Ei}(x) = -\int_x^{\infty} \frac{e^{-t}}{t} dt. \tag{A.21}$$

The logarithmic integral is defined as the Cauchy principal value

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right). \tag{A.22}$$

A.1.6 The cosine and sine integrals

These are defined by

$$\mathrm{Ci}(x) = -\int_x^{\infty} \frac{\cos t}{t} dt \quad \text{and} \quad \mathrm{Si}(x) = \int_0^{x} \frac{\sin t}{t} dt. \tag{A.23}$$

A.1.7 The Riemann zeta-function

The Riemann zeta-function $\zeta(s)$ is a function of a complex variable s. It is defined by the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \tag{A.24}$$

for $\sigma = \mathrm{Re}(s) > 1$. It can also be written as the following product over primes

$$\zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1-p^{-s}} \tag{A.25}$$

for $\sigma > 1$. This is a manifestation of the fundamental theorem of arithmetic which states that every integer can be written in a unique way a product of primes up to the order of the factors. The Riemann zeta-function can be analytically continued to the whole complex plane except at $s = 1$, where it has a simple pole with residue equal to 1. It satisfies the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{A.26}$$

for all values of s.

A.2 Mellin transforms

Mellin's inversion formulae connecting two functions $f(x)$ and $F(s)$ are given

$$F(s) = \int_0^\infty f(x)x^{s-1}dx \quad \text{and} \quad f(x) = \frac{1}{2\pi i}\int_{(\sigma)} F(s)x^{-s}ds. \tag{A.27}$$

The simplest example is $f(x) = e^{-x}$ and $F(s) = \Gamma(s)$ for $\sigma > 0$. Parseval's formula is an identity connecting Mellin transform to their inverses. It can take many forms, some of which are listed below. From [PK01, p. 83, Equation (3.1.13)], we have

$$\frac{1}{2\pi i}\int_{(c)} F(s)G(s)w^{-s}\,ds = \int_0^\infty f(x)g\left(\frac{w}{x}\right)\frac{dx}{x}, \tag{A.28}$$

where $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively. Moreover, if $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$ respectively, and if the line Re $s = c$ lies in the common strip of analyticity of $F(1-s)$ and $G(s)$, then

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi i}\int_{(c)} F(1-s)G(s)\,ds. \tag{A.29}$$

see [PK01, p. 83].

Bibliography

[AB13] G. E. Andrews and B. C. Berndt. *Ramanujan's Lost Notebook, Part* IV. Springer-Verlag, 2013.

[ABYZ04] S. Ahlgren, B. C. Berndt, A. J. Yee, and A. Zaharescu. Integrals of Eisenstein Series and Derivatives of L-functions. *Int Math Res Notices*, (357):303–307, 2004.

[AH12] H. J. Arkfen, G. B. Webber and F. E. Harris. *Mathematical Methods for Physicists, Seventh Edition: A Comprehensive Guide.* Elsevier, 2012.

[Alk12] E. Alkan. Distribution of averages of Ramanujan sums. *Ramanujan J.*, 29(1-3):385–408, 2012.

[AS72] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions.* National Bureau of Standards, 1972.

[Bag81] B Bagchi. *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and other Allied Dirichlet Series.* Calcutta, Indian Statistical Institute, 1981. Ph.D. Thesis.

[Bar91a] K. M. Bartz. On some complex explicit formulae connected with the Möbius function, I. *Acta Arithmetica*, (57):283–293, 1991.

[Bar91b] K. M. Bartz. On some complex explicit formulae connected with the Möbius function, II. *Acta Arithmetica*, (57):295–305, 1991.

[BCHB85] R. Balasubramanian, J. B. Conrey, and D. R. Heath-Brown. Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial. *Journal für die reiene und angewandte Mathematik*, (357):161–181, 1985.

[BCY11] H. M. Bui, B. Conrey, and M. P. Young. More than 41% of the zeros of the zeta function are on the critical line. *Acta Arithmetica*, (150.1):35–64, 2011.

[BD10] B. Berndt and A. Dixit. A transformation formula involving the gamma and Riemann zeta functions in Ramanujan's Lost Notebook. *The Legacy of Alladi Ramakrishnan in the Mathematical Sciences*, pages 199–210, 2010.

[Ber71] B. Berndt. Identities involving coefficients of a class of Dirichlet Series, V. *Trans. Am. Math. Soc.*, (160):139–156, 1971.

[Bha97] R. Bhaskaran. On the versatility of Ramanujan's ideas. *Technical Report 4, Madurai Kamraj University*, pages 118–129, 1997.

[BJ30] H. Bohr and B. Jessen. Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.*, 54(1):1–35, 1930.

[BJ32] H. Bohr and B. Jessen. Über die Werteverteilung der Riemannschen Zetafunktion. *Acta Math.*, 58(1):1–55, 1932.

[Boh05] O. Bohigas. *Compound Nucleus Resonances, Random Matrices, Quantum Chaos*. Recent Perspectives in Random Matrix Theory and Number Theory, Cambridge University Press, 2005.

[BR77] R. Balasubramanian and K. Ramachandra. On the frequency of Titchmarsh's phenomenon for $\zeta(s)$, III. *Proc. Indian Acad. Sci.*, (357):341–351, 1977.

[BT12] D. G. Best and T. S. Trudigan. Linear relations of zeroes of the zeta-function. http://arxiv.org/abs/1209.3843, 2012.

[Bum98] D. Bump. *Automorphic Forms and Representations*. Cambridge University Press, 1998.

[Car] R. D. Carmichael. Expansions of Arithmetical Functions in Infinite Series. *Proc. London Math. Soc.*, S2-34(1):1.

[CG85] J. B. Conrey and A. Ghosh. A simpler proof of Levinson's theorem. *Math. Proc. Cambridge Philos. Soc.*, 97(3):385–395, 1985.

[CG93] J. B. Conrey and A. Ghosh. On the Selberg class of Dirichlet series: small degrees. *Duke Math. J.*, (72):673–693, 1993.

[CK12] T. H. Chan and A. V. Kumchev. On sums of Ramanujan sums. *Acta Arithmetica*, 152(1), 2012.

[CN61] K. Chandrasekharan and R. Narasimhan. Hecke's functional equation and arithmetical identities. *Ann. of Math. (2)*, 74:1–23, 1961.

[Coh49] E. Cohen. An Extension of Ramanujan's sum. *Duke Math. J.*, 16(1), 1949.

[Con83a] J. B. Conrey. Zeros of derivatives of the Riemann's ξ-function on the critical line. *Journal of Number Theory*, (16):49–74, 1983.

[Con83b] J. B. Conrey. Zeros of derivatives of the Riemann's ξ-function on the critical line, II. *Journal of Number Theory*, (17):71–75, 1983.

[Con89] J. B. Conrey. More than two fifths of the zeros of the Riemann zeta function are on the critical line. *Journal für die reiene und angewandte Mathematik*, (399):1–26, 1989.

[Cop35] E. T. Copson. *Theory of Functions of a Complex Variable*. Oxford University Press, Oxford, 1935.

[Cru40] M. M. Crum. On some Dirichlet series. *J. London Math. Soc.*, 15:10–15, 1940.

[CS67] S. Chowla and A. Selberg. On Epstein's Zeta-function. *Journal für die reine und angewandte Mathematik*, 227:86–110, 1967.

[Dav00] H. Davenport. *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2000. Revised and with a preface by H. L. Montgomery.

[dB25] N.G. de Brujin. *Asymptotic Methods in Analysis.* Springer, 1925.

[Del63] H. Delange. On a class of multiplicative arithmetical functions. *Scripta Math.*, 26:121–141 (1963), 1963.

[Del76] H. Delange. On Ramanujan expansions of certain arithmetical functions. *Acta Arith.*, 31(3):259–270, 1976.

[DF34] A. L. Dixon and W. L. Ferrar. Lattice-point Summation Formulae. *Q. J. Math.*, 1:172–185, 1934.

[DF35] A. L. Dixon and W. L. Ferrar. Infinite Integrals of Bessel functions. *Q. J. Math.*, 6:161–174, 1935.

[DI82] J. M. Deshouillers and H. Iwaniec. Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.*, (70):219–288, 1982.

[DI84] J. M. Deshouillers and H. Iwaniec. Power mean values of the Riemann zeta function II. *Acta Arithmetica*, (48):305–312, 1984.

[Dir49] L. Dirichlet. Über die Bestimmung der mittleren Werthe in der Zahlentheorie. *Math. Abhandl. Königl. Akad. Wiss, Berlin*, pages 69–83, 1849.

[Dix12] A. Dixit. Character Analogues of Ramanujan type integrals involving the Riemann Ξ-function. *Pacific Journal of Mathematics*, (225):317–348, 2012.

[Dix13a] A. Dixit. Analogues of the general theta transformation formula. *Proceedings of the Royal Society of Edinburg*, (143A):371–399, 2013.

[Dix13b] A. Dixit. Analogues of the general theta transformation formula. *Proc. Roy. Soc. Edinburgh Sect. A*, 143(2):371–399, 2013.

[Dix13c] A. Dixit. Ramanujan's ingenious method for generating modular-type transformation formulas. *The Legacy of Srinivasa Ramanujan* , 20:163–179, 2013. RMS-Lecture Note Series.

[DKE$^+$00] P. Deligne, D. Kazhdan, P. Etingof, J. W. Morgan, D. S. Freed, D. R. Morrison, L. C. Jeffrey, and E. Witten. *Quantum Fields and Strings: A Course for Mathematicians, Volume 2.* AMS, Institute of Advanced Study, 200.

[DM15] A. Dixit and V Moll. Self-reciprocal functions, powers of the Riemann zeta function and modular-type transformations. *Journal of Number Theory*, (147):211–249, 2015.

[Dow94a] J. S. Dowker. Effective action in spherical domains. *Commun. Math. Phys.*, 162:633–648, 1994.

[Dow94b] J. S. Dowker. Functional determinants on spheres and sectors. *J. Math. Phys.*, 35:4989–4999, 1994.

[DRRZ] A. Dixit, N. Robles, A. Roy, and A. Zaharescu. Koshliakov kernel and identities involving the Riemann zeta function. *Submitted.*

[DRRZ15] A. Dixit, N. Robles, A. Roy, and A. Zaharescu. Zeros of combinations of the Riemann ξ-function on bounded vertical shifts. *Journal of Number Theory*, 149:404–434, 2015.

[DRZ15a] A. Dixit, A. Roy, and A. Zaharescu. Ramanujan-Hardy-Littlewood-Riesz phenomena for Hecke forms. *J. Math. Anal. Appl.*, 426(1):594–611, 2015.

[DRZ15b] A. Dixit, A. Roy, and A. Zaharescu. Riesz-type criteria and theta transformation analogues. *Journal of Number Theory (provisional)*, 2015.

[Dys96] F. Dyson. *Selected Papers of Freeman Dyson with Commentary.* American Mathematical Society, 1996.

[Edw74] H. M. Edwards. *Riemann's zeta function.* Number 58. Academic Press, Pure and Applied Mathematics, 1974.

[EKRW15] E. Elizalde, K. Kirsten, N. Robles, and F. L. Williams. Zeta functions on tori using contour integration. *International Journal of Geometric Methods in Modern Physics*, 12, 2015.

[Eli12] E. Elizalde. *Ten Physical Applications of Spectral Zeta Functions.* Lecture Notes in Physics 855, Springer-Verlag, 2012.

[EOR$^+$94] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini. *Zeta Regularization Techniques with Applications.* World Scientific, Singapore, 1994.

[Eps03] P. Epstein. Zur Theorie allgemeiner Zetafunktionen. *Math. Ann.*, 56:615–644, 1903.

[Eps07] P. Epstein. Zur Theorie allgemeiner Zetafunktionen, II. *Math. Ann.*, 63:205–216, 1907.

[Fen12] S. Feng. Zeros of the Riemann zeta function on the critical line. *Journal of Number Theory*, (132):511–542, 2012.

[Fer35] W. L. Ferrar. Summation formulae and their relation to Dirichlet series. *Compositio Mathematica*, (1):344–360, 1935.

[Fer36] W. L. Ferrar. Some solutions of the equation $F(t) = F(t^{-1})$. *J. London Math. Soc.*, 11(2):99–103, 1936.

[Fer37] W. L. Ferrar. Summation formulae and their relation to Dirichlet series, II. *Compositio Mathematica*, (4):394–405, 1937.

[FMS97] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory.* Springer Verlag, 1997.

[GK91] S. W. Grahan and G. Kolesnik. *Van der Corput's Method of Exponential Sums.* Cambridge Univ. Press, 1991.

[Gla10] M. L. Glasser. Integral representations for the exceptional univariate Lommel functions. *J. Phys. A: Math. Theor.*, (43), 2010.

[GM03] S. S. Gelbart and S. D. Miller. Riemann's zeta function and beyond. *Bulletin (New Series) of the American Mathematical Society*, 41:59–112, 2003.

[Goo78] A. Good. Diskrete Mittel für einige Zetafunktionen. *J. Reine Angew. Math.*, 303/304:51–73, 1978.

[GR07] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products.* Academic Press, 2007.

[Gui47] A. P. Guinand. Some formulae for the Riemann zeta-function. *J. London Math. Soc.*, 22:14–18, 1947.

[Har14] G. H. Hardy. Sur les zéros de la fonction $\zeta(s)$ de Riemann. *Comptes Rendus*, 158:1012–1014, 1914.

[Har15] G. H. Hardy. Note by Mr. G.H. Hardy on the preceding paper. *Quart. J. Math.*, 46:260–261, 1915.

[Har17] G. H. Hardy. Mr S. Ramanujan's mathematical work in England. *J. Indian Math. Soc.*, 9:30–45, 1917.

[Hil84] A. Hildebrand. Über die punktweise Konvergenz von Ramanujan-Entwicklungen zahlentheoretischer Funktionen. *Acta Arith.*, 44(2):109–140, 1984.

[HL18] G. H. Hardy and J. E. Littlewood. Contributions to the Theory of the Riemann Zeta-Function and the Theory of the Distribution of Primes. *Acta Mathematica*, (41):119–186, 1918.

[HT31] G. H. Hardy and E. C. Titchmarsh. Self-reciprocal functions. *Q. J. Math.*, (1):298–309, 1931.

[Hux03] M. N. Huxley. Exponential sums and lattice points. III. *Proc. London Math. Soc.* (3), 87(3):591–609, 2003.

[HW54] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers.* Oxford, at the Clarendon Press, 1954. 3rd ed.

[IK04] H. Iwaniec and E. Kowalski. *Analytic Number Theory*, volume 53. American Mathematical Society, Colloquium Publications, 2004.

[Ing64] A. E. Ingham. *The distribution of prime numbers.* Stechert-Hafner Service Agency, 1964.

[Ivi85] A. Ivic̀. *The theory of the Riemann zeta-function with applications.* John Wiley and Sons, New York, 1985.

[Kir02] K. Kirsten. *Spectral functions in mathematics and physics.* Chapman&Hall/CRC, Boca Raton, FL, 2002.

[Kir10] K. Kirsten. *Functional Determinants in Higher Dimensions via Contour Integrals*, volume 57. A Window into Zeta and Modular Physics, MRSI Publications, 2010.

[KL08] K. Kirsten and P Loya. Computation of determinants using contour integrals. *Am. J. Phys.*, 76:60–64, 2008.

[KM04] K. Kirsten and A. J. McKane. Functional determinants for general Sturm-Liouville problems. *J. Phys. A*, 37:4649–4670, 2004.

[Kos29] N. S. Koshliakov. On Voronoï's sum-formula. *Mess. Math.*, 58:30–32, 1929.

[Kos34] N. S. Koshliakov. Some integral representations of the square of Riemann's function $\Xi(t)$. *Dokl. Akad. Nauk.* , 2:401–405, 1934.

[Kos37] N. S. Koshliakov. On a transformation of definite integrals and its application to the theory of Riemann's function $\zeta(s)$. *Comp. Rend. (Doklady) Acad. Sci. URSS* , 15:3–8, 1937.

[KP99] J. Kaczorowski and A. Perelli. On the structure of the Selberg class, I: $0 \le d \le 1$. *Acta Math.*, 36(182):207–241, 1999.

[KP02] J. Kaczorowski and A. Perelli. On the structure of the Selberg class, V: $1 < d < 5/3$. *Invent. Math.*, 36(150):485–516, 2002.

[KP11] J. Kaczorowski and A. Perelli. On the structure of the Selberg class, VII: $1 < d < 2$. *Annals of Mathematics*, 36(173):1397–1441, 2011.

[KR15] P. Kühn and N. Robles. Explicit Formulas of a generalized Ramanujan sum. *International Journal of Number Theory*, to appear (DOI S1793042116500238), 2015.

[KRR14] P. Kühn, N. Robles, and A. Roy. On a class of functions that satisfies explicit formulae involving the Möbius function. *Ramanujan Journal*, (DOI 10.1007/s11139-014-9608-1), 2014.

[KS99a] N. M. Katz and P. Sarnak. *Frobenius Eigenvalues, and Monodromy*, volume 45. American Mathematical Society Colloquium Publications, 1999.

[KS99b] N. M. Katz and P. Sarnak. Zeros of zeta functions and symmetry. *Bull. Amer. Math. Soc.*, 36(1):1–26, 1999.

[Lam11] Y. Lamzouri. On the distribution of extreme values of zeta and L-functions in the strip $1/2 < \sigma < 1$. *Int. Math. Res. Notices*, 36(23):5449–5503, 2011.

[Lan15] E. Landau. Über die hardysche entdeckung unendlich vieler nullstellen der zetafunktion mit reellem teil 1/2. *Math. Ann.*, 76:212–243, 1915.

[Lev74] N. Levinson. More than One Third of Zeros of Riemann's Zeta-Function are on $\sigma = \frac{1}{2}$. *Advances in Mathematics*, (13):383–436, 1974.

[Lit12] J. E. Littlewood. Quelques conséquences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans le demi-plan $\mathrm{Re}(s) > \frac{1}{2}$. *Comptes Rendus*, (23), 1912.

[Lou79] S. T. Lou. *A lower bound for the number of zeros of Riemann's zeta function on $\sigma = \frac{1}{2}$*, volume 1. Recent Progress in Analytic Number Theory, 1979.

[LR01] L. G. Lucht and K. Reifenrath. Mean-value theorems in arithmetic semigroups. *Acta Math. Hungar.*, 93(1-2):27–57, 2001.

[LRar] X. Li and M. Radziwill. The Riemann zeta-function on vertical arithmetic progression. *Int. Math. Res. Notices*, to appear.

[LS75] B. M. Levitan and I. S. Sargsjan. *Introduction to spectral theory: Selfadjoint ordinary differential operators*, volume 39. Translations of Mathematical Monographs, 1975.

[Mon73] H. L. Montgomery. *The pair correlation of zeros of the zeta function*, volume 24. Proc. Sympos. Pure Math., American Mathematical Society, 1973.

[Mon77] H. L. Montgomery. Extreme values of the Riemann zeta-function. *Comment. Math. Helv.*, (52):511–518, 1977.

[Mot68] Y. Motohashi. A New Proof of the Limit Formula of Kronecker. *Proc. Japan Acad.*, 44:614–616, 1968.

[MP49] S. Minakshisundaram and A. Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math*, 1:242–256, 1949.

[MP99] M. R. Murty and A. Perelli. The pair correlation of zeros of functions in the Selberg class. *Int. Math. Res. Not.*, (10):531–545, 1999.

[MV81] H. L. Montgomery and R. C. Vaughan. The distribution of squarefree numbers. In *Recent progress in analytic number theory, Vol.* 1 *(Durham, 1979)*, pages 247–256. Academic Press, London-New York, 1981.

[MV07] H. L. Montgomery and R. C. Vaughan. *Multiplicative number theory.* I. *Classical theory*, volume 97 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2007.

[MZ02] M. R. Murty and A. Zaharescu. Explicit formulas for the pair correlation of zeros of functions in the Selberg class. *Forum Math.*, (14):65–83, 2002.

[Nak03] M. Nakahara. *Geometry, Topology and Physics.* Graduate Student Series in Physics, Taylor & Francis, 2003.

[NT98] M. Nair and G. Tenenbaum. Short sums of certain arithmetic functions. *Acta Math.*, 180(1):119–144, 1998.

[NYW95] Z. Nan-Yue and K. S. Williams. On the Epstein Zeta Function. *Tamkang Journal of Mathematics*, 26:165–176, 1995.

[Obe74] F. Oberhettinger. *Tables of Mellin transforms.* Springer-Verlag, 1974.

[OM14] F. W. J. Olver and L. C. Maximon. *Digital Library of Mathematical Functions: Chapter 10 Bessel Functions.* National Institute of Standards and Technology, 2014.

[OtR85] A. M. Odlyzko and H. J. J. te Riele. Disproof of the Mertens conjecture. *Journal für die reine und angewandte Mathematik*, (357):138–160, 1985.

[PK01] R. B. Paris and D. Kaminski. *Asymptotics and Mellin-Barnes Integrals*, volume 85 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2001.

[Pol98] J. Polchinski. *String theory, Volume* 1*: An introduction to the bosonic string.* Cambridge University Press, 1998.

[Put54a] C. R. Putnam. On the non-periodicity of the zeros of the Riemann zeta-function. *Amer. J. Math.*, 76:97–99, 1954.

[Put54b] C. R. Putnam. Remarks on periodic sequences and the Riemann zeta-function. *Amer. J. Math.*, 76:828–830, 1954.

[Ram15] S. Ramanujan. New expressions for Riemann's functions $\xi(s)$ and $\Xi(t)$. *Quaterly Journal of Mathematics*, pages 253–260, 1915.

[Ram16] S. Ramanujan. Some formulæ in the analytic theory of numbers. *Messenger Math.*, 45:81–84, 1916.

[Ram18] S. Ramanujan. On certain trigonometrical sums and their applications in the theory of numbers. *Transactions of the Cambridge Philosophical Society*, 22(13), 1918.

[Ram74] K. Ramachandra. On the frequency of Titchmarsh's phenomenon for $\zeta(s)$. *J. London Math. Soc.*, 8(2):683–690, 1974.

[Ram77] K. Ramachandra. On the frequency of Titchmarsh's phenomenon for $\zeta(s)$, II. *Acta Math. Sci. Hung.*, (30):7–13, 1977.

[Ram88] S. Ramanujan. *The Lost Notebook and other Unpublished Papers*. Springer-Verlag, Berlin; Narosa Publishing House, New Delhi, 1988. With an introduction by George E. Andrews.

[Ram00] K. Ramachandra. Notes on the prime number theorem. I. In *Number theory*, Trends Math., pages 351–370. Birkhäuser, Basel, 2000.

[Rie59] B. Riemann. *Üeber die Anzahl der Primzahlen unter einer gegebenen Grösse*. 1859.

[Rie16] M. Riesz. On the Riemann hypothesis. *Acta Mathematica*, (40), 1916.

[RRa] N. Robles and A. Roy. Moments of averages of generalized Ramanujan sums. *Submitted.*

[RRb] N. Robles and A. Roy. Two-parameter generalization of the Selberg formula and the Weil explicit formula. *In preparation.*

[RRZ] N. Robles, A. Roy, and A. Zaharescu. Twisted second moments of the Riemann zeta-function and applications. *Submitted.*

[RS71] D. B. Ray and I. M. Singer. R-torsion and the Laplacian on Riemannian manifolds. *Adv. Math.*, 7:145–210, 1971.

[RS96] Z. Rudnick and P. Sarnak. Zeros of principal L-functions and random matrix theory. *A celebration of John F. Nash Jr., Duke Math.*, (81):269–322, 1996.

[RS03] K. Ramachandra and A. Sankaranarayanan. On an asymptotic formula of Srinivasa Ramanujan. *Acta Arith.*, 109(4):349–357, 2003.

[Sar87] P. Sarnak. Determinants of Laplacians. *Comm. Math. Phys.*, 110:113–120, 1987.

[Sch88] W. Schwarz. Ramanujan expansions of arithmetical functions. In *Ramanujan revisited (Urbana-Champaign, Ill., 1987)*, pages 187–214. Academic Press, Boston, MA, 1988.

[Sel42] A. Selberg. On the zeros of Riemann's zeta-function. *Skr. Norske Vid. Akad. Oslo*, I(10):1–59, 1942.

[Sel92] A. Selberg. *Old and new conjectures and results about a class of Dirichlet series.* Proc. Amalfi Conf. Analytic Number Theory, 1992.

[Shi80a] T. Shintani. A Proof of the Classical Kronecker Limit Formula. *Tokyo J. Math.*, 3:191–199, 1980.

[Shi80b] P. Shiu. A Brun-Titchmarsh theorem for multiplicative functions. *J. Reine Angew. Math.*, 313:161–170, 1980.

[Sie59] C. L. Siegel. *On Advanced Analytic Number Theory.* Tata Institute of Fundamental Research, 1959.

[Sou08] K. Soundararajan. Extreme values of zeta and L-functions. *Math. Ann.*, (342):467–486, 2008.

[SW12] J. Steuding and E. Wegert. The Riemann zeta function on arithmetic progressions. *Exp. Math.*, 21(3):235–240, 2012.

[Tem96] N. M. Temme. *Special Functions: An Introduction to the Classical Functions of Mathematical Physics.* A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1996.

[Tit22] E. C. Titchmarsh. Hankel transforms. *Proc. Camb. Phil. Soc.*, (21):463–473, 1922.

[Tit48] E. C. Titchmarsh. *Introduction to the Theory of Fourier Integrals.* Oxford University Press, 1948.

[Tit68] E. C. Titchmarsh. *The Theory of Functions.* Oxford University Press, second edition, 1968.

[Tit86] E. C. Titchmarsh. *The Theory of the Riemann Zeta-Function.* The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[Var88] I. Vardi. Determinants of Laplacians and Multiple Gamma Functions. *SIAM J. Math. Anal.*, 19:493–507, 1988.

[vdC22] J. G. van der Corput. Verschärfung der Abschätzung beim Teilerproblem. *Math. Ann.*, 87(1-2):39–65, 1922.

[vF05] M. van Frankenhuijsen. Arithmetic progressions of zeros of the Riemann zeta function. *J. Number Theory*, 115(2):360–370, 2005.

[Vor03] G. Voronoi. Sur un problème du calcul des fonctions asymptotiques. *J. Reine Angew. Math.*, 126:241–282, 1903.

[Vor72] S. M. Voronin. The distribution of the nonzero values of the Riemann ζ-function. *Trudy Mat. Inst. Steklov.*, 128:131–150, 260, 1972. Collection of articles dedicated to Academician Ivan Matveevič Vinogradov on his eightieth birthday. II.

[Vor75] S. M. Voronin. A theorem on the "universality" of the Riemann zeta-function. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(3):475–486, 703, 1975.

[Wat31] G.N. Watson. Some self-reciprocal functions. *Quart. J. Math., (Oxford)*, 2:298–309, 1931.

[Wei52] A. Weil. *Sur les "formules explicites" de la théorie des nombres premiers.* 1952.

[Wey12] H. Weyl. Über die Abhangigkeit der Eigenschwingungen einer Membran von deren Begrenzung. *J. Angew. Math.*, 141:1–11, 1912.

[Wil] B. M. Wilson. Proofs of Some Formulae Enunciated by Ramanujan. *Proc. London Math. Soc.*, S2-21(1):235.

[Wil10] F. L. Williams. *Lectures on zeta functions, L-functions and modular forms with some physical applications*, volume 57. A Window into Zeta and Modular Physics, MRSI Publications, 2010.

[Win13] A. Wintner. *Eratosthenian Averages.* Literary Licensing, LLC, 2013.

[Wir61] E. Wirsing. Das asymptotische Verhalten von Summen über multiplikative Funktionen. *Math. Ann.*, 143:75–102, 1961.

[WW62] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis.* Fourth edition. Reprinted. Cambridge University Press, New York, 1962.

[You10] M. P. Young. A short proof of Levinson's theorem. *Arch. Math.*, 95:539–548, 2010.